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MESSENGER OF MATHEMATICS.

FACTORISATION OF $Q=(2^q\mp q)$ AND $(q.2^q\mp 1)$.

By *Lt.-Col. Allan Cunningham, R.E.*, and *Mr. H. J. Woodall, A.R.C.Sc.*

1. *Introduction.* It is proposed in this Paper to study the properties of the four numbers

$$W=2^w-w,\quad Z=2^z+z,\quad Y=y.2^y-1,\quad X=x.2^x+1 \ldots\ldots\ldots(1)$$

[with w, z, y, x all taken positive]

As they have many properties in mutual connexion it will be convenient to study them together. When considered together, they will be denoted collectively by

$$Q=2^q\mp q \quad \text{or} \quad =q.2^q\mp 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1a).$$

2. *Notation and Terminology.* All symbols denote integers.

p denotes an odd *prime*.

ω, Ω denote *odd* numbers; ε, E denote *even* numbers.

q, w, z, y, x; Q, W, Z, Y, X are defined in Art. 1.

q, w, z, y, x are termed the *Roots* of Q, W, Z, Y, X, or of the Congruences (see Art. 7) which show their divisors (p).

$\varpi=p-1$, $\quad P=\xi p$, $\quad \nu\xi=\varpi=p-1$.

$\xi=$ the *Haupt-Exponent* of 2 mod p, *i.e.* the *least exponent* giving $2^\xi\equiv+1 \pmod p$.

L.R. denotes *Least+Residue*.

$K=$L.R. of $2^k \pmod p$; here k is styled the *Radicle* of the Roots q, w, z, y, x which it serves to define (see Art. 8).

$P=\xi p$, the *period* of the roots q, after which they repeat with addition of P (see Art. 8).

Complete Set of Roots (q). The whole set of ξ roots (q) of same kind (*i.e.* w, z, y, or x) in one period from $k=0$ to $k=\xi-1$ (see Art. 8).

Successive Roots (q, q'). Roots (q) of same kind arising from the successive Radicles $k=0$, 1, 2, 3, &c. (see Art. 9).

Consecutive Roots (q, q'). Two roots (q, q') of same kind whose difference $q\sim q'=1$ (see Art. 17, 25*c*).

Cullen's Numbers. $X=x.2^x+1$ (see Art. 28).

3. *Odd and Even Numbers* (Q).

W and Z always *odd*, when $w, z = \omega$; always *even* when $w, z = \varepsilon$...(2*a*).

Y and X always *odd*.....(2*b*).

4. *Algebraic Factorisation.* For a few simple forms of q, algebraic factorisation is possible.

1°. $w, y = \varepsilon^2 = (2\eta)^2$; [*Ex.* $w, y = 4, 16, 32, 36, 64, 100$, &c.].

$$W = (2^2\eta^2)^2 - (2\eta)^2 = P^2 - Q^2 \quad \text{.....(3a)},$$

$$Y = (2\eta)^2 . (2^{2\eta^2})^2 - 1 = P^2 - Q^2 \quad \text{.....(3b)}.$$

2°. $z, x = 4\eta^4$; [*Ex.* $z, x = 4, 64, 324, 1024$, &c.].

$$Z = (2^{\eta^4})^4 + 4\eta^4 \quad = \zeta^4 + 4\eta^4, \text{ a Bin-Aurifeuillian} \quad \text{.....(4a)},$$

$$X = 4\eta^4 . (2^{\eta^4})^4 + 1^4 = 1^4 + 4\zeta^4, \text{ a Bin-Aurifeuillian} \quad \text{.....(4b)}.$$

3°. $q = k^k$, [$k = \omega$]; [*Ex.* $q = 27, 3125$, &c.].

$$W, Z = 2^{k^k} \mp k^k, \quad \text{has the divisor } (2^{k^{k-1}} \mp k) \quad \text{.....(5a)}.$$

$$Y, X = k^k . 2^{k^k} \mp 1, \text{ has the divisor } (k . 2^{k^{k-1}} \mp 1) \quad \text{.....(5b)}.$$

The above numbers all rise rapidly in magnitude with increase of q: complete factorisation is (at present) possible only in the following cases (which are given in Table IV. A, B, at end of this Paper), viz.—

1°. w and $y = 4, 16, 36, 64$; 2°. z and $x = 4, 64$; 3°. $q = 27 = 3^3$.

4a. Aurifeuillians of orders $n = 3, 5, 6$, &c., are also possible as factors of certain of the numbers Q. Such numbers Q are therefore always (algebraically) resolvable into two (or more) factors; but these numbers are all far too large for complete factorisation.

Ex. $q = 12^3, 20^5, 6^6$, &c., define the smallest numbers containing Aurifeuillians of order 3, 5, 6, &c.

5. *Case of* $q = 2^\kappa$. For this form of q, the factorisation of the numbers of the four types Q may be derived from the well-known form $N = (2^n \mp 1)$, some factors of which are known up to high limits of n.

$$W \text{ or } Z = 2^\kappa . (2^{2^\kappa - \kappa} \mp 1), \quad Y \text{ or } X = 2^{2^\kappa + \kappa} \mp 1 \quad \text{.....(6)}.$$

The value of the exponent n is shown in the scheme below up to $\kappa = 8$.

$\kappa =$	1,	2,	3,	4,	5,	6,	7,	8
$n = (2^\kappa - \kappa) =$	1,	2,	5,	12,	27,	58,	121,	248
$n = (2^\kappa + \kappa) =$	3,	6,	11,	20,	37,	70,	135,	264

Complete factorisation is possible (at present) only up to $\kappa=6$; $\kappa=\omega$ gives *n odd*, hence in this case one or more algebraic factors are always present.

Ex. $\kappa=7$ gives $q=128$, $n=135$, $X=128.2^{128}+1=2^{135}+1$.

This is the highest of these numbers (Q) at present completely factorisable. It contains 8 algebraic factors, as shown in the resolution below [by being separated by semi-colons (;)].

$X=3;3;3.19;3.87211;11;331;18837001;811.15121.385838642647891.$

6. 2^{ic} *forms of* Q. The numbers Q are all binomial 2^{ic} forms with determinants D as follows:

$q=\varepsilon$; W and Y have $D=+q$; Z and X have $D=-q$.....$(7a)$;

$q=\omega$; W and Y have $D=+2q$; Z and X have $D=-2q$...$(7b)$;

then, if q or $2q$ have a square factor (say k^2), this may be cancelled from D, thus yielding a reduced determinant $\delta=D\div k^2$.

The value of D, or of the reduced δ, determine the 2^{ic} forms possible to the odd factors of Q ; and the linear forms of factors possible to such numbers may thus be found by the Theory of 2^{ic} forms.

6a. *Simple* 2^{ic} *forms of* Q. For certain simple forms of the roots q the numbers Q take very simple 2^{ic} forms, and their odd prime factors (p) all have simple 2^{ic} forms (the same as those of Q in the simpler cases), and quite simple linear forms. These are shown in the scheme below:

q	*Ex. of* q	W	Z	Y	X
ε^2	4,16,36, 64,100	.	$Z=A^2+B^2$ $p=a^2+b^2$ $=8\varpi\mp1$	.	$X=A^2+B^2$ $p=a^2+b^2$ $=8\varpi\mp1$
ω^2	9,25,49,81	$W=2F^2-E^2$ $p=2f^2-e^2$ $=8\varpi\mp1$	$Z=2D^2+C^2$ $p=2d^2+c^2$ $=8\varpi+1,3$	$Y=2F^2-1^2$ $p=2f^2-e^2$ $=8\varpi\mp1$	$X=2D^2+1^2$ $p=2d^2+c^2$ $=8\varpi+1,3$
$2k^2$	2,8,18,32, 50,72,98	$W=E^2-2F^2$ $p=e^2-2f^2$ $=8\varpi\mp1$	$Z=C^2+2D^2$ $p=c^2+2d^2$ $=8\varpi+1,3$	$Y=2F^2-1^2$ $p=2f^2-e^2$ $=8\varpi\mp1$	$X=2D^2+1^2$ $p=2d^2+c^2$ $=8\varpi+1,3$
$3\varepsilon^2$	12,48,108	$W=A^2-3B^2$ $p=a^2-3b^2$ $=12\varpi\mp1$	$Z=A^2+3B^2$ $p=a^2+3b^2$ $=6\varpi+1$	$Y=3B^2-1^2$ $p=3b^2-a^2$ $=12\varpi\mp1$	$X=3B^2+1^2$ $p=3b^2+a^2$ $=6\varpi+1$
$5\varepsilon^2$	20,80	$W=T^2-5U^2$ $p=t^2-5u^2$ $=10\varpi\mp1$	$Z=T^2+5U^2$ $p=t^2+5u^2$, or $\frac{1}{2}(t^2+5u^2)$ $=20\varpi+1,9;3,7$	$Y=5U^2-1^2$ $p=5u^2-t^2$ $=10\varpi\mp1$	$X=5U^2+1^2$ $p=5u^2+t^2$, or $\frac{1}{2}(5u^2+t^2)$ $=20\varpi+1,9;3,7$

7. *Congruences.* The divisibility of any of the numbers Q by a prime (p), or prime-power (p^k), depends on the solutions (w, z, y, x) of the congruences

$$W=2^w-w\equiv 0,\ Z=2^z+z\equiv 0,\ Y=y.2^y-1\equiv 0,\ X=x.2^x+1\equiv 0 \quad \ldots(\mathrm{mod}\, p,\ \mathrm{or}\ p^k)\ldots(8).$$

The knowledge of the *roots* (w, z, y, x) of these congruences affords the best means of discovering factors (p) of the numbers (Q).

8. *Formulæ for the roots* (q). These can all be made to depend on a relatively small quantity $k\ (<\xi)$, which will be styled the *Radicle* of the roots. Writing (as by Art. 2)

$$K=\text{L.R. of } 2^k \pmod p.$$

Then

$$\left.\begin{aligned} w&=\text{L.R. of } -(K\varpi-kp) \\ z&=\text{L.R. of } +K\varpi+kp \\ y&=\text{L.R. of } -(K\varpi+kp) \\ x&=\text{L.R. of } +(K\varpi-kp) \end{aligned}\right\} \pmod{P=\xi p} \quad \begin{aligned} &\ldots\ldots\ldots\ldots\ldots(8a),\\ &\ldots\ldots\ldots\ldots\ldots(8b),\\ &\ldots\ldots\ldots\ldots\ldots(8c),\\ &\ldots\ldots\ldots\ldots\ldots(8d). \end{aligned}$$

In computing the roots (q) it is often convenient* to rewrite the formulæ in one or other of the following ways:

$$\left.\begin{aligned} w&=\text{L.R. of } -\{(K-k).\varpi-k\} &&=\text{L.R. of } -\{(K-k).p-K\} \\ z&=\text{L.R. of } +\{(K+k).\varpi+k\} &&=\text{L.R. of } +\{(K+k).p-K\} \\ y&=\text{L.R. of } -\{(K+k).\varpi+k\} &&=\text{L.R. of } -\{(K+k).p-K\} \\ x&=\text{L.R. of } +\{(K-k).\varpi-k\} &&=\text{L.R. of } +\{(K-k).p-K\} \end{aligned}\right\} \pmod{P=\xi p} \quad \begin{aligned} &\ldots\ldots(8a'),\\ &\ldots\ldots(8b'),\\ &\ldots\ldots(8c'),\\ &\ldots\ldots(8d'). \end{aligned}$$

From the definition it is evident that

$$w\equiv -x,\quad z\equiv -y,\quad y\equiv -z,\quad x\equiv -w \pmod P \ldots\ldots\ldots\ldots(9),$$

so that—(by a mere change of sign)—w and x may be included in one (modular) formula; and similarly may z and y be included in a single (modular) formula.

The *complete set* of roots ($q \gtrdot P$) of each kind may be computed by taking $k=0,\ 1,\ 2,\ 3,\ \ldots\ (\xi-1)$ in succession, after which they will be found to repeat (in the same order), so that—

There are ξ different roots (q) of each kind $<P$.....(10*a*),

The *period* of each series is $P=\xi p$.....(10*b*),

and the *general values* found by addition of P, $2P$, &c., are

$$(mP+w),\quad (mP+z),\quad (mP+y),\quad (mP+x)\ldots\ldots\ldots\ldots(10c).$$

* The Residue K can then be taken (at sight) from the *Binary Canon* described in Art. 27.

Hence also—

All the roots (q) of each kind in the first set ($k=0$ to $\xi-1$) are $< P$; all those in the second set ($k=\xi$ to $2\xi-1$) are $> P$, but $< 2P$; and so on.
.....(10*d*).

8a. *Succession-formulæ for* w, z, y, x.

Let w_k, z_k, y_k, x_k denote the roots belonging to the radicle k.

Then the formulæ (8*a*—*d*) give at once, mod P,

$$z_{k+1}-z_k \equiv (2^{k+1}-2^k)\varpi+p \equiv 2^k.\varpi+p,$$
$$\equiv z_k-(k-1)p,$$
$$x_{k+1}-x_k \equiv (2^{k+1}-2^k)\varpi-p \equiv 2^k.\varpi-p,$$
$$\equiv x_k+(k-1)p,$$

whence
$$z_{k+1} = \text{L.R. of } 2z_k-(k-1)p \pmod P \text{.....(11}a\text{)},$$
$$x_{k+1} = \text{L.R. of } 2x_k+(k-1)p \pmod P \text{.....(11}b\text{)},$$

and w_{k+1}, y_{k+1} can be inferred from the relation (9).

[These formulæ are convenient for computing successive values q_{k+1}, q_{k+2}, &c., .., from q_k].

9. *Roots* (q) *of same series.* From the definitions it is seen that

$$w \equiv z \equiv +k, \qquad y \equiv x \equiv -k \pmod \xi \text{.....(12}a\text{)},$$
$$w \equiv y \equiv +2^k \equiv K, \quad z \equiv x \equiv -2^k \equiv -K \pmod p \text{.....(12}b\text{)}.$$

Hence the following scheme of the Residues of *successive* roots (q) corresponding to *successive* radicles (k).

$k \equiv$	0,	1,	2,	3, ...	...,	$(\xi-1)$,	
w and $z \equiv$	0,	1,	2,	3, ...	...,	$(\xi-1)$	(mod ξ),
y and $x \equiv$	0,	-1,	-2,	-3, ...	...,	$-(\xi-1)$	(mod ξ),
w and $y \equiv$	2^0,	2^1,	2^2,	2^3, ...	...,	$2^{\xi-1}$	(mod p).
z and $x \equiv$	-2^0,	-2^1,	-2^2,	-2^3, ...	...,	$-2^{\xi-1}$	(mod p).

Thus the set of Residues of each kind is

A complete Arith^c Series in natural order (mod ξ).....(13*a*).

A complete Geom^c Series in natural order (mod p).....(13*b*).

Numerous simple, but important, relations between the roots of the same series, and also between those of different series result from the above properties.

10. *Sums and Products of Roots in pairs.*

Let q, q' be the roots *of same kind* whose radicles are k, k'. Then in each kind of series the following results obtain:

1°. $k+k'=\xi$ gives

$q+q' \equiv 0 \pmod{\xi}$, $qq' \equiv +1 \pmod{p}$ .....(14*a*),

2°. $k-k'=\frac{1}{2}\xi$, $[\xi=\varepsilon]$, gives

$q-q' \equiv 0 \pmod{\frac{1}{2}\xi}$, $q+q' \equiv 0 \pmod{p}$ .....(14*b*),

3°. $k+k' \equiv 0 \pmod{\frac{1}{2}\xi}$, $[\xi=\varepsilon]$, gives

$q+q' \equiv 0 \pmod{\frac{1}{2}\xi}$, $qq' \equiv -1 \pmod{p}$.....(14*c*),

4°. $k=\frac{1}{3}\xi$, $k'=\frac{2}{3}\xi$, $[p=6\varpi+1>7$, $\xi=3i]$, gives

$z+z'=x+x'=P-\varpi$, $w+w'=y+y'=P+\varpi$.....(14*d*).

$qq' \equiv +1 \pmod{p}$..... .....(14*d'*).

11. *Sums and Products of sets of Roots* $[\xi=\mu\kappa]$.

Let $q_0, q_1, q_2, \ldots, q_{k-1}$ be the set of k roots of same kind, whose radicles are $k=0, \kappa, 2\kappa, \ldots, (\mu-1)\kappa$; then

$\Sigma(q) \equiv 0 \pmod{P}$, if $\xi=\omega$), $\equiv 0 \pmod{\frac{1}{2}P}$, if $\xi=\varepsilon$).....(15*a*),

$\Pi(w) \equiv \Pi(y) \equiv \Pi(K) \equiv 2^{\frac{1}{2}\mu(\mu-1)\kappa} \equiv 2^{\frac{1}{2}(\mu-1)\xi} \pmod{p}$

$\equiv -1, \pmod{p}$ if $\mu=\varepsilon$; $\equiv +1, \pmod{p}$ if $\mu=\omega$..(15*b*),

$\Pi(z) \equiv \Pi(x) \equiv \Pi(-K) \equiv (-1)^{\kappa} \cdot 2^{\frac{1}{2}\mu(\mu-1)\kappa} \equiv (-1)^{\kappa} \cdot 2^{\frac{1}{2}(\mu-1)\xi} \pmod{p}$

$\equiv +1, \pmod{p}$ if $\mu=\varepsilon$; $\equiv -1, \pmod{p}$ if $\mu=\omega$..(15*c*).

Hence, taking now the *complete set* of ξ roots (q) of each kind (so that $\mu=\xi$, $\kappa=1$),

$\Sigma(q) \equiv 0 \pmod{P}$, if $\xi=\omega$), $\equiv 0 \pmod{\frac{1}{2}P}$, if $\xi=\varepsilon$).....(15*d*),

$\Pi(w) \equiv \Pi(y) \equiv -1 \pmod{p}$, if $\xi=\varepsilon$), $\equiv +1 \pmod{p}$, if $\xi=\omega$) ...(15*e*),

$\Pi(z) \equiv \Pi(x) \equiv +1 \pmod{p}$, if $\xi=\varepsilon$), $\equiv -1 \pmod{p}$, if $\xi=\omega$) ...(15*f*).

12. *Roots* (q) *of different series* [with same k].

Let w, z, y, x be four roots originating *from the same radicle* (k). Then, from the definition,

$w+x=z+y=*P(=\xi p)$ *always* .....(16).

Hence $\xi=\omega$ gives $(w+x)$ and $(z+y)$ *both odd*, always.....(16*a*),

$\xi=\varepsilon$ gives all q *odd*, when $k=\omega$.....(16*b*),

and all q *even*, when $k=\varepsilon$ .....(16*c*),

$\xi=\varepsilon$ gives the groups w, z, y, x *alternately odd and even*..(16*d*).

Also, the four q are prime to ξ when (and only when) k is prime to ξ .....(17*a*),

And, when k contains a factor (μ) of ξ, then the four q each contain that factor (μ), and not otherwise .....(17*b*).

---

* By this result it suffices to compute *one* of each pair of the roots (w, x), (z, y) direct from the formulæ (8*a*–*d*); the other one can then be obtained by mere subtraction from P.

Hence, if $q_0, q_{\frac{1}{4}}, q_{\frac{1}{2}}, q_{\frac{3}{4}}, q_1$ originate from the radicles $k = 0, \frac{1}{4}\xi, \frac{1}{2}\xi, \frac{3}{4}\xi, \xi$, then $q_0, q_{\frac{1}{4}}, q_{\frac{1}{2}}, q_{\frac{3}{4}}, q_1$ contain $\xi, \frac{1}{4}\xi, \frac{1}{2}\xi, \frac{1}{4}\xi, \xi$ respectively .....(17*c*), and no other roots (q) can contain these factors .....(17*d*).

Also
$$w+y \equiv z+x \equiv 0 \pmod{\xi} \quad \text{.....(18}a\text{)},$$
$$w+z \equiv y+x \equiv 0 \pmod{p} \quad \text{.....(18}b\text{)},$$
$$w-z \equiv y-x \equiv 0 \pmod{\xi} \quad \text{.....(18}c\text{)},$$
$$w-y \equiv z-x \equiv 0 \pmod{p} \quad \text{.....(18}d\text{)}.$$

Some of the above congruences (18*a*–*d*) can sometimes be expressed *as equations* valid for several successive values of k,

From $k=0$ to $2^k \not> \xi$	From $2^{\xi-k+1} \not> \xi$ to $k = \xi - 1$
$z+x = 2K\varpi,\ w+y = P-2K\varpi$	...(18*e*),
$z-x = w-y = 2kp$	$x-z = y-w = 2(\xi-k)\,p$ .....(18*f*),
$w+z = P+2kp,\ y+x = P-2kp$	$w+z = P-2(\xi-k)p,\ y+x = P+2(\xi-k)\,p$ ...(18*g*).

Other *general* relations are below:

$k = \xi - 1$ gives $w+y = P-\varpi,\ z+x = P+\varpi$, *always*.....(19*a*),

$k = \frac{1}{2}\xi - 1$ gives $w+y = P+\varpi,\ z+x = P-\varpi,\ [\xi = \epsilon]$ .....(19*b*),

$k = \frac{1}{4}\xi + 1$ gives $w-y = 2p,\quad z-x = 2p,\quad [\xi = \epsilon]$ .....(19*c*),

$k = \frac{1}{2}(\xi - 1)$ gives $y-w = p,\quad x-z = p,\quad [\xi = \omega > 7]$.....(19*d*),

$k = \frac{1}{2}(\xi + 1)$ gives $w-y = p,\quad z-x = p,\quad [\xi = \omega > 7]$.....(19*e*).

12a. *Roots* (q) *of different series*, [different k].

Let w, x have the radicle k; and let z', y' have the radicle k', where $k \sim k' = \frac{1}{2}\xi$, [$\xi$ even],

Then† $w+y' = \frac{1}{2}P$ or $\frac{3}{2}P,\ z'+x = \frac{3}{2}P$ or $\frac{1}{2}P$.....(20*a*),

and† $w-z' = y'-x = \pm\frac{1}{2}P$.....(20*b*).

13. *Caution* (False Equations). A good deal of care is necessary *in forming equations* with the values of w, z, y, x defined only by the congruences (8*a*–*d*) of Art. 7, mod P; or, with their sums and differences defined only by the congruences of Art. 12, mod ξ or p. Thus, from the former set it *would appear* (at first sight) that the four following sets of results might be *valid equations*.

A	*B*	*C*	*D*
$w-z = y-x =$ L.R. of	$[-2^{k+1}.\varpi \pmod{P}]$	$= w+y,$	
$z-w = x-y =$ L.R. of	$[+2^{k+1}.\varpi \pmod{P}]$	$= z+x,$	
$x-z = y-w =$ L.R. of	$[-2kp \pmod{P}]$	$= y+x,$	
$z-x = w-y =$ L.R. of	$[+2kp \pmod{P}]$	$= w+z.$	

† These Results, along with Results (16), enable the complete set of roots of all the four kinds (w, z, y, x) to be computed from those of any one kind—(when $\xi = \epsilon$)—, by simple subtractions from $\frac{1}{2}P$, P, or $\frac{3}{2}P$.

Now, out of the above sets

$A=B$ ..... are always true.
$A=B=C$..... are true only for certain k.
$C=D$.... are true only for certain k.
$A=B=C=D$ are not true as equations.

The fallacy arises from misuse of the symbol L.R. in the expressions C. The four roots w, z, y, x being defined as the L.R. (*i.e. Least+Residues*) of four quantities say w', z', y', x' mod P (see Art. 7), the expressions C above are *incorrect*. The correct way of writing the *equation* $C=D$ (in the first line of the above) is in fact

$$D=w+y=\text{L.R. of } w'+\text{L.R. of } y' \pmod P.$$

This last expression, the corrected C, is *not the same thing* as the L.R. of $(w'+y')$ the previous expression for C. And similarly in the other cases.

14. *Roots of Fermat's primes.* When the mod (p) is a Fermat's prime $F_n=(2^{2^n}+1)$, so that

$$\xi=2^{n+1},\quad \varpi=2^{\frac{1}{2}\xi},\quad P=\xi p=2^{n+1}.p$$

the congruences $(19a\text{–}d)$ can be converted into *equations* which are valid for a considerable range of the radicle (k).

$k=$	0 to n	$n+1$ to $\frac{1}{2}\xi+n-1$	$\frac{1}{2}\xi+n$	$\frac{1}{2}\xi+n-1$ to $\xi-1$
$w-z=y-x=$	$P-2^{k+1}\varpi$	2^{k+1}	$-2^{k+1-\frac{1}{2}\xi}$	$-2^{k+1-\frac{1}{2}\xi}$
$w+y=$	$2P-2^{k+1}\varpi$	$P+2^{k+1}$	$P-2^{k+1-\frac{1}{2}\xi}$	$P-2^{k+1-\frac{1}{2}\xi}$
$z+x=$	$2^{k+1}\varpi$	$P-2^{k+1}$	$P+2^{k+1-\frac{1}{2}\xi}$	$P+2^{k+1-\frac{1}{2}\xi}$
$x-z=y-w=$	$-2kp$	$P-2kp$	$2P-2kp$	$2kp-P$
$y+x=$	$P-2kp$	$2P-2kp$	$3P-2kp$	$2P-2kp$
$w+z=$	$P+2kp$	$2kp$	$2kp-P$	$2kp$

15. *Various Relations.* Three closely connected relations among the roots q, q' will now be considered.

i. *Equal Roots* $(q=q')$ [in different series].
ii. *Consecutive Roots* $(q-q'=1)$ [in same series].
iii. *Roots with their Doubles* $(q'=2q)$ [in same series].

16. i. *Equal Roots* $(q=q')$. Here three Cases arise, requiring separate treatment.

$1^\circ.\ w=y,\ z=x;\quad 2^\circ.\ w=x,\ z=y;\quad 3^\circ.\ w=z,\ y=x$.....(21).

16a. Case 1°. $w=y,\ z=x$.

The property $w+x=y+z$ [see (16) of Art. 12] shows that either of these equalities involves the other. Now

Let k, k' be the radicles yielding the roots z, x.

By $(8b, d)$ the equality $z=x$ gives

$$\text{L.R. of } [2^{k'}\varpi+k'p]=\text{L.R. of } [2^k.\varpi-kp] \pmod P,$$

therefore $2^{k'}-2^k\equiv 0 \pmod p$, and $k'+k\equiv 0 \pmod \xi$.

Since k', k are both $<\xi$, these involve

$$k'=k=0, \text{ or } =\tfrac{1}{2}\xi \quad \ldots\ldots(22),$$

and, similarly, $w=y$ involve the same $k'=k=0$, or $\frac{1}{2}\xi$.

Hence there are two, and only two, cases of this double equality among the roots (with k, $k'<\xi$), viz.

1°a. $k'=k=0$ gives $z=x=\varpi$, $w=y=P-\varpi$, *always*.....(22a),

and note that, when 2 is a *primitive root* of p, then the latter equality becomes $w=y=\varpi^2$.

1°b. $k'=k=\frac{1}{2}\xi$ [ξ *even*] gives $z=x=\frac{1}{2}P-\varpi$, $w=y=\frac{1}{2}P+\varpi$...(22b).

16b. Case 2°. $w=x$, $z=y$.

Let k, k' be the radicles yielding w, x, respectively.

By (8a, d) the equality $w=x$ gives

$$\text{L.R. of } [P-(2^{k'}\varpi-k'p)]=\text{L.R. of } [2^k\varpi-kp] \pmod P,$$

therefore $2^{k'}+2^k\equiv 0 \pmod p$, and $k'+k\equiv 0 \pmod\xi$.

Since k' and k are both $<\xi$, these involve

$$k'=k=\tfrac{1}{2}\xi, \text{ and } k'+k=\xi,$$

whence $k'=\frac{1}{4}\xi$ and $k=\frac{3}{4}\xi$; or $k'=\frac{3}{4}\xi$ and $k=\frac{1}{4}\xi$.....(23).

Similarly it may be shown that the same k', k lead to $z=y$.

Hence there are two, and only two, cases of this double equality, both requiring $\xi=4i$, as follows:—

For shortness write $\kappa=\frac{1}{4}\xi$, and let q_κ, $q_{3\kappa}$ be the roots arising from the radicles $k=\kappa$, $k=3\kappa$ respectively; then the formulæ (8a, d) give

$$w_\kappa=x_{3\kappa}=\text{L.R. of } [2^{3\kappa}\varpi-3\kappa p],\ w_{3\kappa}=x_\kappa=\text{L.R. of } [2^\kappa\varpi-\kappa p] \pmod P\ldots(23a),$$

$$z_\kappa=y_{3\kappa}=\text{L.R. of } [2^\kappa\varpi+3\kappa p],\ z_{3\kappa}=y_\kappa=\text{L.R. of } [2^{3\kappa}\varpi+3\kappa p] \pmod P\ldots(23b).$$

16c. Case 3°. $w=z$, $y=x$.

These equations are both *impossible*. The proof is quite similar to that of Cases i. and ii., so the detail is not now given.

16d. *Summary of Cases of Equal Roots.*

1°a. $k'=k=0$ gives $z=x=\varpi$, $w=y=P-\varpi$ *always* .....(22a),

1°b. $k'=k=\frac{1}{2}\xi$, [when $\xi=4$] gives $z=x=\frac{1}{2}P-\varpi$, $w=y=\frac{1}{2}P+\varpi$...(22b),

2°. k' or $k=\frac{1}{4}\xi$ and k or $k'=\frac{3}{4}\xi$, [when $\xi=4i$], gives

w, z, y, x (of $k=\frac{1}{4}\xi$) $=x, y, z, w$ (of $k=\frac{3}{4}\xi$) respectively...(23),

and there are *no other cases* $<P$; but these cases recur once in each period of P, *i.e.* from $q=mP$ to $(m+1)P$.

17. ii. *Consecutive Roots* $(q'-q=1)$ [of same kind].

Two Cases arise requiring separate treatment.

$1^\circ.\ w-w'=1,\ x'-x=1;\quad 2^\circ.\ z'-z=1,\ y-y'=1.$

17a. Case 1°. Let k', k be the radicles yielding the roots x', x.

Then, by (8*d*), the equality $x'-x=1$ gives

L.R. of $[2^{k'}\varpi-k'p)]$ − L.R. of $[2^k\varpi-kp]=1 \pmod P$.

Hence $k'-k\equiv -1 \pmod\xi$, and $2^{k'}-2^k\equiv -1 \pmod p$.

Since k', k are both $<\xi$, these require $k'=0$, $k=1$.....(24).

These give $x'=\varpi$, $x=\varpi-1$; $w'=P-\varpi$, $w=P-\varpi+1$ always...(24*a*).

17b. Case 2°. Let k', k be the radicles yielding the roots z, z'.

Then, by (8*b*), the equality $z'-z=1$ gives

L.R. of $[2^{k'}\varpi+k'p]$ − L.R. of $[2^k\varpi+kp]=1 \pmod P$.

Hence $k'-k\equiv +1 \pmod\xi$, and $2^{k'}-2^k\equiv -1 \pmod p$.

Since k', k are both $<\xi$, these require $k'=\frac{1}{2}\xi+1$, $k=\frac{1}{2}\xi$...(25).

These require $\xi=\epsilon$, and give

$z=\frac{1}{2}P-\varpi,\ z'=\frac{1}{2}P-\varpi+1;\ y=\frac{1}{2}P+\varpi,\ y'=\frac{1}{2}P+\varpi-1$ ...(25*a*).

The above (24*a*, 25*a*) are the only possible cases $(<\xi)$ of $q'-q=1$.

18. iii. *Roots and their Doubles* $(q'=2q)$ [of same kind].

Let k', k be the radicles yielding the roots x', x.

Then, by (8*d*), the equality $x'=2x$ gives

L.R. of $[2^{k'}\varpi-k'p]=2\times$ L.R. of $[2^k\varpi-kp] \pmod P$.

Hence $2^{k'}-2^{k+1}\equiv 0 \pmod p$, and $k'-2k\equiv 0 \pmod\xi$.

Since $k<\xi$, these require $k'=2k$, whence $2^{2k}-2^{k+1}\equiv 0 \pmod p$.

Hence $2^{k-1}-1\equiv 0 \pmod p$, which (with $k<\xi$) involves $k=1$, $k'=2$..(26).

The cases of $w'=2w$, $z'=2z$, $y'=2y$ lead to the same radicles (k, k').

These give the following results, and no other cases exist (with $k<\xi$).

$w=P-\varpi+1$, $w'=2w$ (always $>P$, but $<2P$ .....(26*a*),

$z=3\varpi+1$, $z'=2z$ ($>P$, when $p=5, 7$).....(26*b*),

$y=P-3\varpi-1$, $y'=2y$ ($>P$, except when $p=5, 7$).....(26*c*),

$x=\varpi-1$, $x'=2x$ ($<P$, except when $p=3$) .....(26*d*).

Thus w' and one of z', y' (usually z') always fall in the second set (*i.e.* between P and $2P$).

19. *Sums of* W, Z, Y, X.

Summing from $w, z, y, x=1$ to w, z, y, x,

$$\Sigma_1^w W=\Sigma_1^w(2^w-w)=2(2^w-1)-\tfrac{1}{2}w(w+1)\ldots\ldots\ldots\ldots(27a),$$

$$\Sigma_1^z Z=\Sigma_1^z(2^z+z)=2(2^z-1)+\tfrac{1}{2}z(z+1)\ldots\ldots\ldots\ldots(27b).$$

Noting that Z contains p when $z=p-1$, and X contains p when $x=p-1$, and when $x=p-2$, it follows that the 5 quantities

$$\Sigma_1^{p-1}W,\ \Sigma_1^{p-1}Z,\ Z_{p-1},\ X_{p-2},\ X_{p-1}\ \text{all contain } p\ldots\ldots\ldots(27c).$$

Next, to sum Y, X, note that by known rules

$$\Sigma_1^x x\,.\,2^x=(x+1)\,.\,2^{x+1}-2^{x+2}+2.$$

Hence

$$\Sigma_1^y Y=(y+1)\,.\,2^{y+1}-2^{y+2}+2-y=Y_{y+1}-Z_{y+2}+5\ldots\ldots\ldots(27d),$$

$$\Sigma_1^x X=(x+1)\,.\,2^{x+1}-2^{x+2}+2+x=X_{x+1}-W_{x+2}-1\ldots\ldots\ldots(27e).$$

20. *Algebraic and Semi-Algebraic Roots* (q).

Here follow two Tables A, B of cases wherein the roots (q) are given—under the conditions quoted—by simple algebraic formulæ, not merely by congruences as before in Results $(8a$–$d)$.

21. *Algebraic Roots* (Tab. A). The "conditions" in this Table are all simple and definite, and the radicles (k) yielding the roots (q) are all simple and definite.

Algebraic Roots (q). Tab. A.

Condition.	k	w	z	y	x
Always	0	$\varpi^2\equiv P-\varpi$	ϖ	$\varpi^2\equiv P-\varpi$	ϖ
	1	$P-\varpi+1$	$3\varpi+1$	$P-3\varpi-1$	$\varpi-1$
	2	$P-x$	$6\varpi+2$	$P-z$	$2\varpi-2$
	3	$P-x$	$11\varpi+3$	$P-z$	$5\varpi-3$
	4	$P-x$	$20\varpi+4$	$P-z$	$12\varpi-4$
	k	$P-x$	$(K+k)\varpi+k^*$	$P-z$	$(K-k)\varpi-k^*$
$\xi=\varepsilon$	$\frac{1}{2}\xi$	$\frac{1}{2}P+\varpi$	$\frac{1}{2}P-\varpi$	$\frac{1}{2}P+\varpi$	$\frac{1}{2}P-\varpi$
$\xi=4i$	$\frac{1}{4}\xi$	$2^{\frac{3}{4}\xi}\varpi-\frac{3}{4}P^*$	$2^{\frac{1}{4}\xi}\varpi+\frac{1}{4}P^*$	$2^{\frac{3}{4}\xi}\varpi+\frac{3}{4}P^*$	$2^{\frac{1}{4}\xi}\varpi-\frac{1}{4}P^*$
$\xi=4i$	$\frac{3}{4}\xi$	$2^{\frac{1}{4}\xi}\varpi-\frac{1}{4}P^*$	$2^{\frac{3}{4}\xi}\varpi+\frac{3}{4}P^*$	$2^{\frac{1}{4}\xi}\varpi+\frac{1}{4}P^*$	$2^{\frac{3}{4}\xi}\varpi-\frac{3}{4}P^*$
$(2/p)_2=-1$	$\frac{1}{2}\xi-1$	$P-x$	$P-y$	$p+\frac{1}{2}\varpi$	$p-\frac{1}{2}\varpi$
$(2/p)_2=+1$	$\xi-1$			$p-\frac{1}{2}\varpi$	$p+\frac{1}{2}\varpi$
$(2/p)_4=-1$	$\frac{1}{2}\xi-2$			$2p+\frac{1}{4}\varpi$	$2p-\frac{1}{4}\varpi$
$(2/p)_4=+1$	$\xi-2$			$2p-\frac{1}{4}\varpi$	$2p+\frac{1}{4}\varpi$
$(2/p)_8=-1$	$\frac{1}{2}\xi-3$			$3p+\frac{1}{8}\varpi$	$3p-\frac{1}{8}\varpi$
$(2/p)_8=+1$	$\xi-3$			$3p-\frac{1}{8}\varpi$	$3p+\frac{1}{8}\varpi$
$(2/p)e=-1$	$\frac{1}{2}\xi-\kappa$		$[e=2^\kappa]$	$\kappa p+\varpi/e$	$\kappa p-\varpi/e$
$(2/p)e=+1$	$\xi-\kappa$		$[e=2\kappa]$	$\kappa p-\varpi/e$	$\kappa p+\varpi/e$

* The L.R. of these quantities to be taken mod P.

22. *Semi-Algebraic Roots* (TAB. B.). The radicles (k) in this Table are defined by conditions of form

$$K \mp k = m\xi - \lambda \text{ or } m\xi + \lambda \text{ ; or } = \omega.\tfrac{1}{2}\xi - \lambda \text{ or } \omega.\tfrac{1}{2}\xi + \lambda \quad \ldots\ldots(28),$$

wherein m, ω, λ have the following limits

$$m, \omega \text{ may be } \pm \text{ ; and } m \not> \nu, \ \omega \not> 2\nu \quad \ldots\ldots(28a),$$

$$\lambda \text{ is } +, \text{ and } < \xi \text{ when with } m, \text{ or } < \tfrac{1}{2}\xi \text{ when with } \omega \quad \ldots\ldots(28b).$$

The formulæ for q in the Table are of course only special cases of the general formulæ ($8a'$–$8d'$). They show at once that

The magnitudes of w, z, y, x depend essentially on the Residue $|\lambda|$ of $(K \mp k) \bmod \xi$ ..... (29).

This is well seen in the formulæ

$$w = \lambda p + K \text{ when } K - k = m\xi - \lambda \quad \ldots\ldots(29a),$$

$$z = \lambda p - K \quad ,, \quad K + k = m\xi + \lambda \quad \ldots\ldots(29b),$$

$$y = \lambda p + K \quad ,, \quad K + k = m\xi - \lambda \quad \ldots\ldots(29c),$$

$$x = \lambda p - K \quad ,, \quad K - k = m\xi + \lambda \quad \ldots\ldots(29d).$$

Hence, for any particular value of λ, one or more of the four conditions (28) may have no solutions at all, or only one solution (k, K), or several solutions (k_1, K_1), (k_2, K_2), (k_3, K_3), &c. Each solution gives one value of the root (w, z, y, x) concerned, whose magnitude is determined by λ in such a way that

$$w \text{ and } y \text{ are } > \lambda p, \text{ but } < (\lambda + 1)p; \ z \text{ and } x \text{ are } > (\lambda - 1)p, \text{ but } < \lambda p \ldots (30)$$

Attaching—for distinction's sake—the suffixes w, z, y, x to the symbols m and λ which belong to the roots w, z, y, x respectively, it is seen that

With the same radicle (k)
$$\begin{cases} \lambda_w + \lambda_x = \lambda_z + \lambda_y = \xi \quad \ldots\ldots(31a), \\ m_w - m_x = m_y - m_z = 1 \quad \ldots\ldots(31b). \end{cases}$$

The formulæ (29a–d) show that

$$K > k \text{ requires } m_w \text{ to be } + \text{ and } > 0 \quad \ldots\ldots(32a),$$

$$K < k \text{ requires } m_x \text{ to be } - \text{ and } |m| > 0 \quad \ldots\ldots(32b),$$

$$\text{In all cases } m_y \text{ is } + \text{ and } > 0 \quad \ldots\ldots(32c).$$

The four lines at the foot of Table B give values of z, y depending on the simple conditions

$$K + k = \text{one of } p - 2, p - 1, p, p + 1 \quad \ldots\ldots(33).$$

These are merely special forms of the general condition (28) given here on account of their simplicity.

Semi-Algebraic Roots (q). TAB. B.

$K-k$	w	x	$K+k$	z	y
$m.\xi-\lambda$	$\lambda p+K$	$P-w$	$m.\xi-\lambda$	$P-y$	$\lambda p+K$
$m.\xi-1$	$p+K$	$P-w$	$m.\xi-1$	$P-y$	$p+K$
$m.\xi$	K	$P-K$	$m.\xi$	$P-K$	K
$m.\xi+1$	$P-x$	$p-K$	$m.\xi+1$	$p-K$	$P-z$
$m.\xi+\lambda$	$P-x$	$\lambda p-K$	$m.\xi+\lambda$	$\lambda p-K$	$P-z$
$\omega.\frac{1}{2}\xi-\lambda$	$\frac{1}{2}P+\lambda p+K$	$\frac{1}{2}P-\lambda p-K$	$\omega.\frac{1}{2}\xi-\lambda$	$\frac{1}{2}P-\lambda p-K$	$\frac{1}{2}P+\lambda p+K$
$\omega.\frac{1}{2}\xi-1$	$\frac{1}{2}P+p+K$	$\frac{1}{2}P-p-K$	$\omega.\frac{1}{2}\xi-1$	$\frac{1}{2}P-p-K$	$\frac{1}{2}P+p+K$
$\omega.\frac{1}{2}\xi$	$\frac{1}{2}P+K$	$\frac{1}{2}P-K$	$\omega.\frac{1}{2}\xi$	$\frac{1}{2}P-K$	$\frac{1}{2}P+K$
$\omega.\frac{1}{2}\xi+1$	$\frac{1}{2}P-p+K$	$\frac{1}{2}P+p-K$	$\omega.\frac{1}{2}\xi+1$	$\frac{1}{2}P+p-K$	$\frac{1}{2}P-p+K$
$\omega.\frac{1}{2}\xi+\lambda$	$\frac{1}{2}P-\lambda p+K$	$\frac{1}{2}P+\lambda p-K$	$\omega.\frac{1}{2}\xi+\lambda$	$\frac{1}{2}P+\lambda p-K$	$\frac{1}{2}P-\lambda p+K$
			$p-2$	$P-y$	$p+K$
			$p-1$	$P-K$	K
			p	k	$P-k$
			$p+1$	$w+k$	$P-x$

22a. *Roots* (q) *within given limits.* Let the limits assigned be as in (30), viz.

w or y between λp and $(\lambda+1)p$; z or x between $(\lambda-1)p$ and λp.

Then the solutions* (k, K) of the appropriate one of the equations $(29a-d)$

$$K\mp k=m\xi-\lambda \text{ or } =m\xi+\lambda$$

give the required radicle (k) with which the roots (q) may be computed from the appropriate formulæ $(29a-d)$ of

$$q=\lambda p\mp K.$$

23. *Small roots* q $(<p)$. For purposes of factorisation of the numbers Q of this Memoir it is important to be able to find the small roots $q<p$ independently. This is really a particular case of the last Article, the Results $(29a-d)$ being modified as follows:

$\lambda=0$ gives $w=K$, when $K-k=m\xi$ .....(34*a*),

$\lambda=1$ gives $z=p-K$, when $K+k=m\xi+1$ .....(34*b*),

whence $z=k$, when $K+k=p$ .....(34*c*),

$\lambda=0$ gives $y=K$ when $K+k=m\xi$ .....(34*d*),

$\lambda=1$ gives $x=p-K$ when $K-k=m\xi+1$ .....(34*e*).

All the cases of small roots $(q<p)$ are included in the above Results $(34a-d)$, and it is seen that

The existence of small roots $(q<p)$ requires the solvability of the proper* equation $K\mp k=m\xi$ or $m\xi+1$ .....(34).

* The direct solution of these equations is in general extremely difficult without the help of suitable Tables such as the *Binary Canon*, as explained in Art. 27.

In a few special cases the above Results reduce to algebraic forms *for all primes alike* (as in Tab. A), viz.

$k=0$ gives $K=1$, whence $z=p-1$, $x=p-1$, *always* .....(35*a*),

$k=1$ gives $K=2$, whence $x=p-2$, *always* .....(35*b*),

$k=\frac{1}{2}\xi-1$ gives $K=\frac{1}{2}(p-1)$, whence $x=\frac{1}{2}(p+1)$, when $(2/p)_2=-1$...(35*c*),

$k=\xi-1$ gives $K=\frac{1}{2}(p+1)$, whence $y=\frac{1}{2}(p+1)$, when $(2/p)_2=+1$...(35*d*).

23a. *Small roots, simple case.*

The principal difficulty in finding the roots (q) of a given prime (p) is the necessity that the symbol K in the formulæ ($8a'-d'$) for q should be the *Least+Residue* of $2^k \bmod p$, whilst the labour of computing this Residue is considerable when k is large. This difficulty *does not occur* in the case of primes $p>2^k$, because in this case $K=2^k$ and therefore requires no reduction.

Writing therefore $K=2^k$ in equations ($34a-d$), the small roots $q<p$ (in the case of such primes) are given by

$w=2^k$ when $2^k-k=m.\xi$ .....(36*a*),

$z=p-2^k$ when $2^k+k-1=m.\xi$ .....(36*b*),

$y=2^k$ when $2^k+k=m.\xi$ .....(36*c*),

$x=p-2^k$ when $2^k-k-1=m.\xi$ .....(36*d*).

The cases of $w=2^k$, $y=2^k$ are sufficiently met in Art. 5, so may be passed by; those of z, x are further developed below.

For shortness let $\phi(k)=2^k\pm k-1$.

Compute the value of $\phi(k)$ for each form for successive values of $k=2, 3, 4, 5$, &c.

Resolve each value of $\phi(k)$ in every possible way into a pair of cofactors; denote these by m, ξ, so that $\phi(k)=m.\xi$.

For each value of ξ find the prime p which gives ξ as the Haupt-Exponent of the Base 2, so that $2^\xi \equiv +1 \pmod p$.

Neglect all cases in which $p<2^k$.

Finally, in all the cases wherein $p>2^k$, the primes p possess the roots $z=p-2^k$, and $x=p-2^k$ given by formulæ (36*b*, *d*).

23b. *Examples* (Tabs. C.).

These two Tables give a number of examples of the above process of finding small roots (z, x) of certain primes.

Col. 2 of each Table shows the value of $(2^k\pm k-1)$ resolved into two cofactors ($m.\xi$).

Col. 3 shows the prime p for each ξ, such that $2^\xi \equiv +1 \pmod p$.

Col. 4 shows the root z or $x=p-2^k$ resulting for that prime.

The factorisations $m.\xi$ shown are limited to those for which the mod p is known, and is $>2^k$, but $<10^6$.

The process above described is of very limited application for following reasons:—

1°. The difficulty of factorising $(2^k\pm k-1)$ when $k>23$.

2°. The primes (p) corresponding to the factors (ξ) of $(2^k\pm k-1)$ are known for only a limited number of values of ξ.

3°. The values of z, x resulting are in most cases too large for practical factorisation of Z, X.

TAB. C.

	$2^k+k-1=m\xi$					$2^k-k-1=m\xi$			
k	m	ξ	$p>2^k$	$z=p-2^k$	k	m	ξ	$p>2^k$	$x=p-2^k$
2	1	5	31	27	4	1	11	23	7
3	2	5	31	23	4	1	11	89	73
3	1	10	11	3	5	2	13	8191	8159
4	1	19	524287	524271	5	1	26	2731	2699
5	4	9	73	41	6	3	19	524287	524223
5	1	36	37	5	6	1	57	32377	32313
6	3	23	178481	178417	7	8	15	151	23
8	1	263	23671	23415	7	5	24	241	113
9	40	13	8191	7079	7	4	30	331	203
9	20	26	2731	2219	7	3	40	61681	61553
9	13	40	61681	61169	7	2	60	1321	1193
9	10	52	1613	1101	8	19	13	8191	7935
9	5	104	858001	857489	8	13	19	524287	524031
9	4	130	409891	409379	8	1	247	15809	15553
9	2	260	521	9	9	2	251	54217	53705
9	2	260	51481	50969	10	1	1013	6079	5055
9	1	520	42641	42129	11	1	2036	4073	2025
11	49	42	5419	3371	12	3	1361	8167	4071
13	14	586	26371	18179	15	1424	23	178481	145713
13	14	586	33403	25211	17	5698	23	178481	47409
14	863	19	524287	507903					
15	37	886	48731	15963					

24. *Radicle* (k) *of a given root* (q). The formulæ (29*a*–*d*) of Art. 22 enable the Problem of finding the radicle (k) of a given root (q) of a given prime (p) to be solved.

STEP i.

w or y given. Solve for λ, K in $\lambda p+K=w$ or y.....(37*a*),

z or x given. Solve for λ, K in $\lambda p-K=z$ or x.....(37*b*).

Since λ, K are both +, and $\lambda<\xi$, and $K<p$, one (unique) solution exists for each, and is easily found.

STEP ii. With λ, K now known, solve for m, k* as follows:

* Note that this Step is the finding of k from the known K (the Residue of $2^k \bmod p$), which is in general a difficult matter, but here rendered easy by the help of the known λ.

For w; $m\xi+k=K+\lambda$ .....(38a),

For z; $m\xi-k=K-\lambda$ .....(38b),

For y; $m\xi-k=K+\lambda$ .....(38c),

For x; $m\xi+k=K-\lambda$ .....(38d).

Since m is $\not> \nu$, and k is $+$ and $< \xi$, one (unique) and easily found solution exists for each.

Examples. These have been purposely chosen with roots w, z, y, x arising from the same radicle (k), so as to illustrate the relations between the four λ's and between the four m's, and to show that m_x is $-$ when $K<k$.

Ex. 1°. $p=53$; $\xi=52$, $\nu=1$; here 2 is a primitive root of p.

Data.	λ, K	$m\xi\pm k=K\pm\lambda$	m, k
$w=383=\lambda p+K$	7, 12	$m\xi+k=12+\ 7=\ 19$	0, 19
$z=1631=\lambda p-K$	31, 12	$m\xi-k=12-\ 31=-19$	0, 19
$y=1125=\lambda p+K$	21, 12	$m\xi-k=12+\ 21=\ 33$	1, 19
$x=2373=\lambda p-K$	45, 12	$m\xi+k=12-\ 45=-33$	-1, 19

Ex. 2°. $p=113$; $\xi=28$, $\nu=4$; here 2 is a 4-tic Residue of p.

Data.	λ, K	$m\xi\pm k=K\pm\lambda$	m, k
$w=2256=\lambda p+K$	19, 109	$m\xi+k=109+19=128$	4, 16
$z=1360=\lambda p-K$	13, 109	$m\xi-k=109-13=96$	4, 16
$y=1804=\lambda p+K$	15, 109	$m\xi-k=109+15=124$	5, 16
$x=908\ =\lambda p-K$	9, 109	$m\xi+k=109-9\ =100$	3, 16

24a. *Radicles (k) of small given roots ($q<p$).*

These are much easier to find than those of the most general case of Art. 24, and are at the same time much the more useful for factorisation purposes. The general Rules (38a–d) of Art. 24 take the simpler forms below.

Noting that k, K, m, λ are all $+$, and $k<\xi$, $K<p$, $m\not>\nu$, and λ now $=1$,

	2 a prime root of p			$\xi=(p-1)\div\nu$, $\nu>1$			
Given	K	k	*Condition.*	K	k	*Condition.*	m
w	w,	w,	$K-k=0$	w,	$K-m\xi$	$K-k=m\xi$,	max. (38a'),
z	$p-z$,	z,	$K+k=p$	$p-z$,	$m\xi-k+1$,	$K+k=m+1$,	min. (38b'),
y	y,	$\varpi-y$,	$K+k=\varpi$	y,	$m\xi-y$,	$K+k=m\xi$,	min. (38c'),
x	$p-x$,	$\xi-x$,	$K-k=1$	$p-x$,	$(\nu-m)\xi-x$,	$K-k=m\xi+1$,	max. (38d').

[m should be the max. or min. integer making w, z, y, x, $+$].

25. *Composite modulus* (m).

Let a, b be two unequal odd primes, and let $m=ab$.

Let ξ_a, ξ_b, ξ_m be the Haupt-Exponents of the Base 2.

Let q_a, q_b, q_m be "roots" *of the same kind.*

Let P_a, P_b, P_m be the "periods" of the roots q_a, q_b, q_m.

Let n_a, n_b, n_m be the *number* of roots q_a, q_b, q_m in a period.

Let a, b, m, as subscripts, indicate the moduli in use with the symbols ξ, q, P, n.

Each of the roots q_a, q_b generates a whole series of roots of the general forms $(\lambda.P_a+q_a)$, $(\mu.P_b+q_b)$, one in each successive period P_a, P_b respectively, so that a root (q_m) falling in both series is given by

$$q_m=\lambda.P_a+q_a=\mu.P_b+q_b \quad \ldots\ldots(39),$$

wherein P_a, P_b, q_a, q_b are *given*, and λ, μ are to be found by solving the indeterminate equation

$$\lambda.P_a-\mu.P_b=q_b-q_a \quad [\lambda,\ \mu \text{ to be } \textit{integers}] \quad \ldots\ldots(40).$$

The least integer values of λ, μ will give a root q_m in the first period, *i.e.* $<P_m$.

The roots of all four kinds (w_m, z_m, y_m, x_m) will evidently repeat, and *in the same order*, in each period given by

$$P_m=\text{L.C.M. of } P_a,\ P_b=\text{L.C.M. of } a\xi_a,\ b\xi_b \quad \ldots\ldots(41).$$

The maximum and minimum values of P_m are

$$P_m(\text{max.})=a\xi_a.b\xi_b \text{ when } a\xi_a \text{ is prime to } b\xi_b \quad \ldots\ldots(41a),$$

$$P_m(\text{min.})=b\xi_b \text{ when } \xi_b \text{ contains } a\xi_a \quad \ldots\ldots(41b).$$

Similarly, if $m=abc$, [a, b, c different odd primes], then

$$P_m=\text{L.C.M. of } P_a,\ P_b,\ P_c=\text{L.C.M. of } a\xi_a,\ b\xi_b,\ c\xi_c \quad \ldots\ldots(42).$$

Ex. Take $a=3$, $b=5$, $c=7$; then $\xi_a=2$, $\xi_b=4$, $\xi_c=3$;

$$P_{15}=60,\ P_{21}=42,\ P_{35}=420;\ P_{105}=420.$$

25a. *Number of roots* (q_m) in period P_m.

The *number* of the above equations (40) for determining λ, μ is evidently = the product $(n_a.n_b)$ of the numbers of the roots q_a, q_b in their periods.

When P_a is prime to P_b, these equations are all solvable in integers (λ, μ), so that the number of roots (q_m) in the period P_m is then, $n_m=n_a.n_b$ (the maximum value).

But, if P_a, P_b contain a common factor (ξ), the equation (40) will be solvable in integers (λ, μ) only when (q_b-q_a) also contains that factor, so that n_m is then always $<n_a.n_b$.

The general *Rule is

$$n_m=P_m \div ab=(\text{L.C.M. of } a\xi_a,\ b\xi_b) \div ab \quad \ldots\ldots(43).$$

* It does not seem easy to give a proof of this in any short compass; but its correctness can be readily tested by example.

The values of n_m under various relations between $a\xi_a$, $b\xi_b$ are shewn in the Table below, with numerical examples in small primes a, b. The maximum and minimum values are

n_m (max.) $= \xi_a . \xi_b$, when $a\xi_a$ is prime to $b\xi_b$ .....(43*a*),

n_m (min.) $= \xi_a . \xi_b$, as shown in Table .....(43*b*).

The minimum is seen to be remarkably small compared with the possible maximum.

Mutual primes.	*Relation.*	n_m	*Ex.* 1°. m	n_m	*Ex.* 2°. m	n_m
$(a\xi_a,\ b\xi_b)$		$\xi_a . \xi_b$	5.7;	12	7.11;	30
$(a,\ \xi_b),\ (b,\ \xi_a)$	$\xi_a = \xi_b$	ξ_b	23.89;	11	29.113;	28
	$\beta\xi_a = \xi_b$	ξ_b	3.11;	10	5.13;	12
	$\xi_a = \alpha f,\ \xi_b = \beta f$	$\alpha\beta f$	13.19;	36	11.23;	10
$(b,\ \xi_a),\ (\xi_a,\ \xi_b)$	$a = \xi_b$	ξ_a	3.7;	2	5.31;	4
	$\beta a = \xi_b$	$\beta\xi_a$	7.29;	12	7.43;	6
$(b,\ \xi_a)$	$\beta\xi_a = \xi_b$	ξ_a	5.41;	4	31.311;	5
	$\beta a\xi_a = \xi_b$	$\beta\xi_a$	3.13;	4	3.19;	6

25b. *Relations of roots* (q_m).

From the insolvability of many of the equations (40), whenever $a\xi_a$, $b\xi_b$ have a common factor, it follows that the relations among the roots (q_m) with composite modulus must be quite different from those obtaining with a prime modulus. It must suffice to illustrate this by one example.

25c. *Consecutive roots* $(q'_m - q_m = 1)$.

Let the roots q'_m, q_m arise from the combination of the pairs of roots $(q'_a,\ q'_b)$, $(q_a,\ q_b)$ with moduli a, b. It is evident then that $(q'_a,\ q_a)$, $(q'_b,\ q_b)$ must be consecutive roots in the periods of moduli a, b. Hence the existence of consecutive roots $(q'_m,\ q_m)$ requires the existence of consecutive roots in the series of both a, b.

It will be convenient to consider the roots of types w_m, z_m, y_m, x_m separately.

Case 1°, (*Type* x_m). The consecutive roots $(x',\ x)$ for moduli a, b are (see Art. 17*a*)

$$x'_a = a-1,\ x_a = a-2;\quad x'_b = b-1,\ x_b = b-2,$$

and the equations for x'_m, x_m are

$x'_m = \lambda' . P_a + x'_a = \mu' . P_b + x'_b$.....(44*a*),

$x_m = \lambda . P_a + x_a = \mu . P_b + x_b$ .....(44*b*),

and, the equations for determining λ', μ', λ, μ, are

$\lambda' . P_a - \mu' . P_b = x'_b - x'_a = b - a$.....(45*a*),

$\lambda . P_a - \mu . P_b = x_b - x_a = b - a$.....(45*b*),

whence $\lambda'=\lambda$, $\mu'=\mu$; *i.e.* the *conditions* (45*a*, *b*) yielding x'_m, x_m are *identical*, so that x'_m x_m exist together, or else neither exists. When λ, μ have been found, x'_m, x_m are given by (44*a*, *b*).

Noting that $(b-a)$ is prime to both a, b, the solvable and insolvable cases are as shown below.

Solvable
1°. If P_a is prime to P_b .....(46*a*),
2°. If the G.C.M. of ξ_a, ξ_b is a factor of $(b-a)$, and a prime to ξ_b, and b prime to ξ_a.....(46*b*),

Insolvable
1°. If (a, ξ_b) have a common factor, or (b, ξ_a) .....(46*c*),
2°. If the G.C.M. of ξ_a, ξ_b is a non-factor of $(b-a)$.....(46*d*).
[This case seems impossible if a prime to ξ_b & b prime to ξ_a]

Case 2°, (*Type* w_m). When x'_m, x_m exist, w'_m, w_m may be found from the fundamental relation

$$w'_m+x'_m=P_m=w_m+x_m.$$

Case 3°, (*Type* z_m). The *consecutive* roots z', z for moduli a, b are (see Art. 17*b*)

$$z'_a=\tfrac{1}{2}P_a-a+2,\ z_a=\tfrac{1}{2}P_a-a+1 \quad .....(47a),$$
$$z'_b=\tfrac{1}{2}P_b-b+2,\ z_b=\tfrac{1}{2}P_b-b+1 \quad .....(47b),$$

whence the equations for λ', μ', λ, μ are

$$\lambda'.P_a-\mu'.P_b=z'_b-z'_a=\tfrac{1}{2}(P_b-P_a)-(b-a) \quad .....(48a),$$
$$\lambda.P_a-\mu.P_a=z_b-z_c=\tfrac{1}{2}(P_b-P_a)-(b-a) \quad .....(48b),$$

whence $\lambda'=\lambda$, μ', μ, *i.e.* the conditions (48*a*, *b*) for z'_m, z_m are *identical*, so that z'_m, z_m exist together, or else neither exists. When λ, μ have been found, z'_m, z_m may be found from equations similar to (44*a*, *b*).

The condition (48*a*) for λ, μ are somewhat stricter than in Case 1°, as z', z exist only when ξ is *even*, (see Art. 17*b*). The conditions of solvability are summarised below.

Ex. The Table below shows examples of the principles of this Article.

Mutual primes.	*Condition.* [*F means factor*]	*Data.* a, b	ξ_a, ξ_b	x,z	q'_a, q_a	q'_b, q_b	q'_m, q_m
$(a\xi_a, b\xi_b)$	·	5, 7	4, 3	x	4, 3	6, 5	45, 44
				z	7, 6	. .	. .
(a, ξ_b), (b, ξ_a)	F of ξ_a, ξ_b in (q_b-q_a)	5,13	4,12	x	4, 3	12,11	324, 323
				z	7, 6	67,66	847, 846
(a, ξ_b), (b, ξ_a)	F of (ξ_a, ξ_b) in (q_b-q_a)	13,19	12,18	x	12,11	18,17	1728,1727
	F of (ξ_a, ξ_b) not in (q_b-q_a)	13,19	12,18	z	67,66	154,153	. .
(b, ξ_a)	$\xi_b=af$	5,11	4,10	x	4, 3	10, 9	. .
				z	7, 6	46,45	. .

Solvable
1°. If P_a is prime to P_b, and ξ_a. ξ_b *both even* .....(49*a*),
2°. If the G.C.M. of ξ_a, ξ_b is a factor of $(z_b \sim z_a)$, and ξ_a, ξ_b *both even*, and *prime* to b, a respectively...(49*b*).

Unsolvable
1°. If either of ξ_a ξ_b is *odd* .....(49*c*),
2°. If (a, ξ_b) or (b, ξ_a) have a common factor .....(49*d*),
3°. If one of z_a, z_b is *even*, and the other *odd* .....(49*e*),
4°. If the G.C.M. of ξ_a, ξ_b is a non-factor of $(z_a \sim z_b)$...(49*f*).

Case 4°, (*Type* y_m). When z'_m, z_m exist, y'_m, y_m may be found from the fundamental relation

$$z'_m + y'_m = P_m = z_m + y_m.$$

26. *Power-Moduli* (p^k).

Let q_a denote a root of $Q_a \equiv 0 \pmod{p^a}$.

This involves that q_a is a root of $Q_a \equiv 0 \pmod{p, p^2, p^3, \ldots, p^{a-1}}$. Hence, if P_1, P_2, P_3, ..., P_a be the periods of q_1, q_2, q_3, ..., q_a, then

$$q_2 = m_2 P_1 + q_1, \quad q_3 = m_3 P_2 + q_2 = m'_3 P_1 + q_1, \quad q_4 = \&c. \ldots\ldots (50).$$

Hence each of the ξ roots q_1 of p gives rise to one (and only one) root q_2, q_3, ..., q_a, [each $q_a <$ its own P_a], for each power p^2, p^3, ..., p^a of p, for

$$q_a = q_{a-1} \equiv \ldots\ldots \equiv q_3 \equiv q_2 \equiv q_1 \pmod{P_1} \ldots\ldots\ldots\ldots (51),$$

so that

There are ξ roots q_a in the "period" P_a to each power p^a of p...(51*a*).

It remains to show how to find the coefficients of m_a. Take q_2 to denote either of the roots w_2, z_2 of p^2, so that

$$Q_2 = (W_2 \text{ or } Z_2) = 2q_2 \mp q_2 \equiv 0 \pmod{p^2} \ldots\ldots\ldots\ldots (52),$$

and $q_2 = m_2 P_1 + q_1$

therefore $Q_2 = 2^{m_2 P_1 + q_1} \mp (m_2 P_1 + q_1)$

$$\equiv (2^{q_1} \mp q_1) \mp m_2 \xi p \pmod{p^2}$$

because $2^{P_1} = 2^{\xi p} \equiv +1 \pmod{p^2}$ .....(52*a*).

Dividing by p, and noting that $Q_1 = W_1$ or Z_1 is divisible by p,

$$m_2 \xi \equiv \pm Q_1 \div p \pmod{p} \ldots\ldots\ldots\ldots (53),$$

and m_2 can be determined from this congruence, as w_1 and z_1 (which determine Q_1) are supposed known.

Similarly it may be shown that

$$m_3 \xi \equiv \pm Q_2 \div p \pmod{p}, \quad m_4 \xi \equiv \pm Q_3 \div p \pmod{p}, \ \&c. \ldots\ldots (54),$$

whence m_3, m_4, &c., may be determined, and thence q_3, q_4, &c., in succession. [Note that, in these latter congruences, the + sign belongs to w, and the − sign to z].

The roots y_a, x_a of $Y_a\equiv 0$, $X_a\equiv 0 \pmod{p^a}$ may be found by a similar process; but, more simply (when z_a, w_a are known) from the relation

$$w_a+x_a=\xi p^a=z_a+y_a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(55).$$

27. *Use of *Binary Canon.*

This Table gives *at sight* the *Least + Residues*, (say K), of all powers (as 2^k) of the Base 2, from $k=0$ to $k=\xi$, for all prime and prime-power moduli p and $p^k \not> 1000$, (with the exponent k as Argument); and gives also conversely the exponent (k) of the power 2^k of 2 whose Least + Residue is K (which is the Argument of this latter Table).

This Canon is of especial †help in connexion with the roots w, z, y, x of all primes and prime-powers $\not> 1000$, as follows:

1°. It enables any, or all, (the complete set), of the roots (q) of each kind to be computed direct from the formulæ ($8a'-a'$) of Art. 8 without much labour, (because it gives the value of K at sight).

2°. It enables the solutions (k, K) of the conditions ($3\S a-d$)

$$K\mp k=m\xi-\lambda \text{ or } =m\xi+\lambda$$

to be picked out without much labour; hence—

2°*a*. It enables all the roots of (q) of each kind within given limits, say $(\lambda-1)p$ to $(\lambda+1)p$, to be readily found by Art. 22*a*.

2°*b*. It enables all the small roots ($q<p$) of each kind to be readily found.

27a. *Binary Canon‡ Extension.*

This Table gives *at sight* the Least Residues, both + and −, (say K and K'), of every power, as 2^k of the Base 2 up to $k=100$, for every prime and prime-power modulus p and $p^k \not> 10000$ (with the exponent k as Argument).

This Table gives the same facilities as the previous one for all primes and prime-powers p and $p^k \not> 10000$, through the Ranges of $k=1$ to 100 (using the + Residues K), and $k=\frac{1}{2}\xi$ to $\frac{1}{2}\xi+100$, (using the − Residues K' of $2^{k-\frac{1}{2}\xi}$, when $\xi=\epsilon$).

28. *Cullen's§ Numbers*, $(X=x.2^x+1)$. These numbers have two properties worth special notice—

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* *Binary Canon*, by Lt.-Col. Allan Cunningham, R.E., London (1900).

† The labour of the work detailed in pars 1°, 2°, 2°*a*, 2°*b* on any considerable scale would be prohibitive without this help.

‡ *Binary Canon Extension*, by the present joint authors; (at present only in MS.).

§ Attention was first called to these numbers (X) by Father J. Cullen, S.J., in the *Educational Times* of Dec. 1905, Question 15897.

i. Frequency of *small* "Least Divisors."

ii. Rarity of primes.

28a. *Small "Least Divisors."* A thorough search has been made for small *divisors of *all* these numbers (X) in the range $x=1$ to 1000. The result is exhibited in three ways below:—

[Note that L.F. means "Least Factor" of X].

1°. **L.F.>100.** [Range $x=1$ to $x=99$].

$x=$ 53, 75 } Only 2 cases of L.F.>100.
L.F.=5591, 149 }

2°. **L.F.>1000.** [Range $x=1$ to 1000].

$x=53,$ 141, 233, 245. Range $x=8$ to 250

L.F.=5591, $>10^4$, $>10^3$, $>10^3$. Only 4 cases of L.F.>1000

$x=748,$ 765,
L.F.=2971, 1559,

$x=251,$ 252, 285, 293, 318, 333, 402, 412, 435, 436, 453, 473;
516, 532, 533, 579, 580, 581, 587, 588, 609, 634, 648, 693, 711, 713;
753, 778, 816, 849, 869, 870, 899, 900, 915, 916, 921, 933, 941, 942, 953;

L.F. unknown in these 41 cases, but >1000.

Abstract	$x=1$ to 250	251 to 500	501 to 750	751 to 1000	1 to 1000
L.F.>1000	4 cases	12 cases	15 cases	16 cases	47 cases

This Abstract shows well the efficiency of the small primes ($p<1000$) as divisors of X; this is most marked up to $x=250$, beyond which they are far less efficacious.

3°. **L.F.>$(x+2)$.** [Range $x=1$ to 1000].

$x=33,$ 53, 75, 113, 141, 159, 219, 233, 234, 245,

L.F.=47, 5591, 149, 241, $>10^4$, 317, 587, $>10^3$, 467, $>10^3$,

$x=250$ to 388	389, 393	394 to 470	471	472 to 500
L.F. as in Case 2°	491, 823	As in Case 2°	941	As in Case 2°
$x=501$ to 747	748	749 to 764	765	766 to 1000
L.F. as in Case 2°	2971	As in Case 2°	1559	As in Case 2°

Abstract	$x=1$ to 250	251 to 500	501 to 750	751 to 1000	1 to 1000
L.F.<$(x+2)$	10 cases	15 cases	15 cases	16 cases	56 cases

This Abstract shows the remarkable efficiency of the small primes $p \not> (x+2)$ as divisors of the numbers X.

28b. *Rarity† of primes.* Here $x=1$ gives $X=3$, which is the *only known* prime among these numbers X. All these

* This search is *complete* only for divisors $p<1000$: from $p>1000$ up to 10000 it is complete only to the limit $k=100$ of the Binary Canon Extension (Art. 27*a*) used for solving the Congruence (36*d*).

† This property was pointed out by Father Cullen in the *Educational Times*, Question quoted above. He stated that all the X were composite from $x=2$ to $x=100$ with a possible exception when $x=53$. The range of composites was extended to $x=200$ (including $x=53$, but excluding $x=141$) by Col. A. Cunningham (see *Mathl. Questions and Solutions from the Educational Times*, vol. x., 1906, Solution of Q. 15897, pages 45–47), and has since been extended to 232 (still excluding 141) by the present joint authors.

X are composite from $x=2$ to $x=232$ (see Tab. IV.) with the possible exception of $X_{141}=141.2^{141}+1$, for which no divisor has yet been found, (none exists <10300).

This rarity of primes seems to be partly accounted for by the following causes:—

1°. The very efficient distribution of the small divisors, (see Art. 28*a*).

2°. By the two algebraic roots $x=(p-1)$, $(p-2)$ of every odd prime being such that every one of the latter is a divisor of a special pair of the X, which are *different to the pair* similarly related to every other prime, so that the first consecutive odd primes are divisors of $2n$ *different* numbers X.

3°. When $(2/p)_2=-1$, then $x=\frac{1}{2}(p+1)$; and when $(2/p)_2=+1$, then $x=\frac{1}{2}(3p-1)$ give p a divisor of X, (see Tab. A, Art. 21); and these two numbers X will often differ from each other and from the two cited in para. 2° above.

Ex. Take $x=141$ (cited above). There are 33 odd primes <141; these give 66 composite X due to $x=(p-1)$, $(p-2)$. Also there are 15 primes $p=8i+3$, and 16 primes $p=8i+5$ which make $(2/p)_2=-1$, and give $x=\frac{1}{2}(p+1)<141$; and there are 6 primes $p=8i+7$, and 4 primes $p=8i+1$ which make $(2/p)_2=+1$, and give $x=\frac{1}{2}(3p-1)<141$; thus 41 primes making $(2/p)_2=\mp 1$ are divisors of the X from $x=1$ to 140; but these give only 18 new cases [*i.e.* different from each other, and from those due to $x=(p-1)$, $(p-2)$]. Thus the algebraic roots quoted in paras 2°, 3° above account for $66+18=84$ of the 139 composite X from $x=2$ to $x=140$.

29. *Divisors of* X_x.

The following are general properties of the divisors (p) of X_x:—

Since $-x.2^x \equiv +1 \pmod{p}$

$-x$ must be a 2^{ic} Residue of every p of X_x, if $x=\varepsilon$.....(56*a*),

$-2x$,, ,, ,, ,, if $x=\omega$.....(56*b*).

This property serves to exclude one-half of the total number of primes $<\sqrt{X_x}$ from the list of possible divisors.

The two following are *necessary* and *sufficient* conditions that p may be a divisor—

1°. The congruence $2^\alpha \equiv -x \pmod{p}$ must be *possible* ....(57*a*).

2°. And $\alpha+x$ must $=\xi$, or $m.\xi$ .....(57*b*).

The conditions (57*a*) may be looked on as the *preliminary* condition for all possible divisors, and condition (57*b*) as the special condition.

The following variant of the above (57*a*, *b*) is often useful.

1°*a*. The congruence $2^{\alpha'} \equiv +x' \pmod{p}$ must be *possible* $[x=\varepsilon]$....(57*a'*).

2°*a*. And $\alpha'+x'$ must $=\frac{1}{2}\xi$, or $\omega.\frac{1}{2}\xi$ ....(57*b'*).

The conditions (34*e*), (36*d*) of Art. 23, 23*a*

$K-k-1=\xi$, or $m.\xi$ .....(58*a*),

$2^k-k-1=\xi$, or $m.\xi$ .....(58*b*),

are really special cases of the above, convenient for use with the *Binary Canon* (Art. 27), or with its *Extension* (Art. 27*a*).

30. *Divisors of* X_{141}. The long sequence of 231 composite numbers from $x=2$ to 232, with the possible exception of X_{141}, seemed so remarkable that an extended search has been made for divisors of this number. The various methods employed are explained in the following Art. 30*a*–*d*.

30a. *Linear Form of Factors* (p) *of* X_{141}.

Here $x=141$, so that the 2^{ic} condition for the divisors of X_{141} is

$$(-282/p)=+1, \text{ so that } (-47/p)(6/p)=+1 \quad \ldots\ldots(60).$$

Now $(-47/p)=(-1/p)(47/p)=+(p/47)$ always, so that

$$(p/47) \text{ and } (6/p) \text{ should both } =+1, \text{ or both } =-1 \quad \ldots\ldots(60a).$$

Writing p in the twin forms

$$p=47i+r=24i'+r' \quad \ldots\ldots(60b),$$

the values of r, r' giving $(p/47)=\pm 1$ and $(6/p)=\pm 1$ are shown in the Table below.

$(p/47)$	p	$(6/p)$	r'
$+1$	1, 2, 3, 4, 6, 7, 8, 9, 12, 14, 16, 17 18, 21, 24, 25, 27, 28, 32, 34, 36, 37, 42	$+1$	1, 5, 19, 23
-1	5, 10, 11, 13, 15, 19, 20, 22, 23, 26, 29, 30 31, 33, 35, 40, 41, 43, 44, 45, 46, 38, 39	-1	7, 11, 13, 17

The values of p given by the two formulæ (60*b*) together are the only ones that can possibly be divisors of X_{141}. They serve to exclude one-half of the total number of primes $p<\sqrt{X_{141}}$.

30b. *Direct trial of divisors* (p) *of* X_{141}.

Let $R=$ Least + Residue of $2^{141} \pmod p$.

$$\text{Then } 141.R \text{ should } \equiv -1 \pmod p \quad \ldots\ldots(61).$$

Direct trial of all primes $p \not> 10631$ has been made by this process without success. This shows that

$$\text{The Least divisor } (p) \text{ of } X_{141} \text{ is } > 10300 \quad \ldots\ldots(62).$$

30c. *Congruential Trial.*

By Art. 29 the fundamental condition that p may be a divisor of x is that—

$$\text{The congruence } 2^{\alpha} \equiv -141 \pmod p \text{ must be possible} \ldots\ldots(63).$$

It is obviously always possible in two cases.

(1). When $\xi=p-1$; (2). When $\xi=\frac{1}{2}(p-1)$ and $(-282/p)=+1$.

In other cases it is not easy to predict the possibility, or impossibility, of the congruence *à priori*.

When the value of α is known, or can be easily found, this gives the easiest method of all of testing whether the prime p is, or is not, a divisor of X_{141}, by the simple Rule (57*b*)

$$\alpha+141=\xi \text{ or } m.\xi \quad \ldots\ldots(64).$$

The labor of solving the congruence (63) for a *given prime* (>1000) is in general prohibitive; but, by reversing the process, (*i.e.* taking α as given), a limited number of primes (p) can be found for which the congruence, or a modified form of it, is obviously satisfied (for a given α); thus—

Let the two series of numbers

$$A=2^{\alpha}+141, \quad B=2^{\beta}-141,$$

be factorised into prime factors for a series of successive values of α, β respectively, so that

$$A=a_1a_2a_3\ldots, \quad B=b_1b_2b_3\ldots.$$

Then the primes of types a, b are such that

$$2^{\alpha}\equiv -141 \pmod{a_1, a_2, \text{&c.}}, \quad 2^{\beta}\equiv +141 \pmod{b_1, b_2, \text{&c.}} \quad \ldots(65).$$

Here the exponents α, β are all *known* for the particular primes of types a, b respectively. And the final Tests of divisibility of X_{141} by any of them are (see Art. 29)

$$\alpha+141=\xi \text{ or } m\xi, \quad \beta+141=\tfrac{1}{2}\xi \text{ or } \omega.\tfrac{1}{2}\xi, \quad [\xi=\varepsilon] \ldots\ldots\ldots(66).$$

The process is of very limited applicability for two reasons :—

1°. The difficulty of factorising A, B into primes when $\alpha, \beta > 30$,

2°. Ignorance of the ξ for primes $a, b > 10^5$.

The process was actually tried by factorising A, B as far as was found possible up to $\alpha, \beta \not> 40$, and γ, without success; in fact the values of ξ, $\frac{1}{2}\xi$ were in all cases far too large for the Tests above.

30d. *Use of the radicles* (k) *of trial divisors* (p).

From Result (62) it follows that $x=141$ must be much less than even the Least divisor of X_{141}, so that the simple Rule (38*d'*) of Art. 24*a* applies, giving

$$K=p-141, \quad k=\mu.\xi-141 \ldots\ldots\ldots(67),$$

where μ is the least integer giving $k+$.

These give at once for a given trial prime (p)—(when its ξ is known)—the radicle (k) that would yield the root $x=141$. The Test would still have to be applied that

$$2^k \text{ should } \equiv +K \pmod{p} \ldots\ldots\ldots(68a),$$

or,

$$2^{k_0} \text{ should } \equiv 141, \text{ where } k_0=k-\tfrac{1}{2}\xi, \text{ [and } \xi=\varepsilon] \ldots\ldots(68b).$$

To use these formulæ, a Table of the values of K, k for successive values of ξ (from $\xi=17^*$ upwards) was formed, and the primes (p) corresponding taken from the known factorisations of $(2^{\xi} \mp 1)$. This secures all possible *small* values of ξ, k_1, k_0. There is no advantage in pushing this process beyond k or $k_0=141$, as the direct process of Art. 30*b* is then easier.

31. *Primes of form X.* It is (at present) uncertain whether any primes (>3) exist of this form, $X=(x.2^x+1)$. Such primes would have interesting properties. These include

$$(2/X)_2=+1, \text{ and } (x/X)_2=+1, \textit{ always} \ldots\ldots(69a),$$

$$(2/X)_8=+1, \text{ if } x=\varepsilon^2 \ldots\ldots(69b),$$

$$(2x/X)_4=+1, \text{ if } x=\omega \ldots\ldots(70a),$$

$$(2x/X)_8=+1, \text{ if } x=4\alpha+1 \ldots\ldots(70b),$$

$$(2x/X)_{2\varepsilon}=+1, \text{ if } x=\varepsilon.\alpha+1, \text{ when } \varepsilon=2^{\lambda} \ldots\ldots(70c),$$

$$(x/X)_x=+1, \quad \text{and } x^{\frac{1}{2}x}\equiv +1 \pmod{x} \textit{ always} \ldots\ldots(71),$$

Since $(x/X)_2=1$, therefore $y^{2^{x-1}}\equiv +1 \pmod{x}$ *always*.....(71*a*),

Hence X is a divisor of $(x^{2^{x-1}}-1$ *always*) .....(71*b*),

and is therefore a divisor of *one* of the Algebraic Factors thereof, *i.e.* of

$$(x^{2^{x-2}}+1)\,(x^{2^{x-3}}+1)\,.\,(x^{2^{x-4}}+1)\ \&c.\ \ldots\ldots\ldots\ldots\ldots(71c).$$

This suggests that one of the simplest Tests of the primality of any X for a given value of x would be to compute the Residues of

$$x^{2^t},\ x^{2^{t+1}},\ x^{2^{t+2}},\ \&c.,\ \ldots\ (\mathrm{mod}\ X),$$

starting from $x^{2^t} > X$, up to $x^{2^{x-2}}$, (if necessary). If X be prime, some one of these must $\equiv -1\ (\mathrm{mod}\ X)$. The work would be very laborious (as the lowest $x = 141$).

Primes of form X seem likely as possible divisors of Fermat's Numbers $F_n = (2^{2^n}+1)$. The necessary and sufficient condition is

$$(2/X)_x = +1,\ \text{which gives}\ 2^{2^x} \equiv +1\ldots\ldots\ldots\ldots\ldots\ldots(72),$$

whereby X would be a divisor of some Fermat's Number F_n where $n \gtrless x-2$, and may be $< x-2$.

32. *Tables.* At the end of this Paper follow Tables of two sorts:—

Tab. IA–E, II., III., of roots (q).

Tab. IVA, IVB, of factorisation of W, Z, Y, X.

Tab. VA, VB of small factors of X.

32a. *Table of roots* (q). These Tables are of two sorts:—

Tab. IA–E, of complete sets of Roots (q).

Tab. II., III., of small roots (q).

32b. *Tab. I*A–E. These Tables give the complete sets of ξ roots of each kind w, z, y, x for each of the small primes (p) and prime-powers (p^k) stated below:—

$p = 3, 5, 7, \ldots\ldots, 47$; also $p = 73, 89, 127, 257$;

$p^k = 9, 27, 81, 243, 729$; $25, 125, 625$; $49, 343$; $121, 169, 289$.

These Tables are arranged under order of the radicle (k) from $k = 0$ to $k = (\xi - 1)$ as Argument. This arrangement has the advantage of enabling many of the properties (Art. 8–18) depending on the radicle (k) to be easily recognised. Such complete Tables are valuable for tracing out the properties of the roots; but they are so lengthy, and the roots (q) run into such high numbers that space does not admit of printing many of them.

32c. *Tab. II.* This Table gives *all* the small roots $w, z, y, x \not> 100$, and also *all* the $x > 100$ up to $\not> 250$, for every prime from $p=43$ to 199, and prime-powers $p^k=49$, 81, 121, 125.

32d. *Tab. III.* This Table gives *all* the small roots (w, z, y, x) under the limits named below, and for the selection of primes $(p > 200)$ named:—

$w, z \not> 100$, for *all* primes > 200, but $\not> 10000$, *complete*;

$y \not> 100$, $x \not> 250$ for *all* primes > 200, but $\not> 1000$, *complete*;

$y \not> 100$, $x \not> 250$ for *selected* primes only > 1000, but $\not> 10000$;

but, in the column of roots x, the two algebraic roots $x=(p-1)$, $(p-2)$,—(being easily recognised)—are *omitted* (to save space).

In this Table all primes (p) having no roots (q) within the limits named are omitted (to save space). The Table is *complete* in w, z $(\not> 100)$ up to $p \not> 10000$, and in y, x $(\not> 100)$ up to $p \not> 1000$* only.

32e. *Tab. IV*A, B. *Factorisation Tables.* These give the factorisation of W, Z, Y, X into prime factors, made as complete as possible with the means available, for all values of q up to $q=66$.

The search for factors is complete as follows:—

W, Z; All factors $p \not> 10^4$ cast out.

Y, X; All factors $p \not> 10^3$ cast out.

In certain cases of Y, X, when expressible as binary trinomials of form $(2^m \mp 2^n \mp 1)$, the search for factors has been extended to $p \not> 10^4$. These cases are marked ‡ in the Table. In this Table a semi-colon (;) on the right indicates that the factorisation (into prime factors) is *complete*. A query (?) on the right indicates that the composition of the last large factor is unknown.

32f. *Tab. V*A, B. *Small Factors of X.* This Table gives small *factors* of $X=(x.2^x+1)$ from $x=60$ to 260; there is no attempt at completeness. Sufficient (small) factors (p) are shown to illustrate the properties stated in Art. 28–28*b*. Special search for factors was made in three cases.

$x=233, 245$; the sign † means "No factors < 1000."

$x=141$; the sign ‡ means "No factors < 10000."

---

* The roots y, x are so much more difficult to find than w, z that it was found impracticable to continue the (complete) search for roots (q) of primes $p > 1000$, (this being the limit of the complete Binary Canon, Art. 27*a*).

TAB. Ia.

k		0	1		0	1	2	3		0	1	2			
w	**3**	4	5	**5**	16	17	14	3	**7**	15	16	11			
z		2	1		4	13	6	7		6	19	17			
y	$\xi 2$	4	5	$\xi 4$	16	7	14	13	$\xi 3$	15	2	4			
x	*P*6	2	1	*P*20	4	3	6	17	*P*21	6	5	10			

k		0	1	2	3	4	5	6	7	8	9
w	**11**	100	101	92	63	104	65	86	7	58	39
z		10	31	62	3	94	45	46	37	8	49
y	$\xi 10$	100	79	48	107	16	65	64	73	102	61
x	*P*110	10	9	18	47	6	45	24	103	52	71

k		0	1	2	3	4	5	6	7	8	9
w		144	145	134	99	16	149	90	115	152	57
z		12	37	74	135	88	137	66	67	56	21
y		144	119	82	21	68	19	90	89	100	135
x		12	11	22	57	140	7	66	41	4	99

	13														
k		10	11		0	1	2	3	4	5	6	7			
w	$\xi 12$	10	59	**17**	120	121	106	59	84	117	30	111			
z	*P*156	94	71		16	49	98	43	52	53	38	127			
y		62	85	$\xi 8$	120	87	38	93	84	83	98	9			
x		146	97	*P*136	16	15	30	77	52	19	106	25			

k		0	1	2	3	4	5	6	7	8
w		324	325	308	255	130	203	330	223	332
z		18	55	110	201	22	329	240	43	314
y		324	287	232	141	320	13	102	299	28
x		18	17	34	87	212	139	12	119	10

	19									
k		9	10	11	12	13	14	15	16	17
w	$\xi 18$	189	226	281	30	193	158	69	214	143
z	*P*342	153	154	137	84	301	32	159	52	161
y		189	188	205	258	41	310	183	290	181
x		153	116	61	312	149	184	273	128	199

k		0	1	2	3	4	5	6	7	8	9	10
w	**23**	231	232	211	146	246	170	248	128	118	75	219
z		22	67	134	245	191	60	28	194	250	86	241
y	$\xi 11$	231	186	119	8	62	193	225	59	3	167	12
x	*P*253	22	21	42	107	7	83	5	125	135	178	34

TAB. Ib.

k		0	1	2	3	4	5	6	7	8	9	10
w		784	785	758	675	480	61	6	679	372	541	38
z		28	85	170	311	564	229	342	539	92	793	542
y		784	727	642	501	248	583	470	273	720	19	270
x		28	27	54	137	332	751	806	133	440	271	774
	29											
k	$\xi 28$	11	12	13	14	15	16	17	18	19	20	21
w	$P812$	627	152	797	434	491	576	717	158	635	748	133
z		11	544	769	378	379	352	269	74	467	412	273
y		801	268	43	434	433	460	543	738	345	400	539
x		185	660	15	378	321	236	95	654	177	64	679

k	22	23	24	25	26	27		0	1	2	3	4
w	498	387	136	417	138	363	**31**	125	126	97	8	109
z	778	135	444	221	558	391		30	91	27	23	139
y	34	677	368	591	254	421	$\xi 5$	125	64	128	132	16
x	314	425	676	395	674	449	$P155$	30	29	58	147	46

k		0	1	2	3	4	5	6	7	8
w		1296	1297	1262	1155	904	365	582	979	404
z		36	109	218	399	724	5	1194	871	188
y		1296	1223	1114	933	608	1327	138	461	1144
x		36	35	70	177	428	967	750	353	928
k		9	10	11	12	13	14	15	16	17
w		549	802	1271	840	1273	770	1059	268	1313
z		117	1270	875	48	1021	266	51	916	1277
y		1215	62	457	1284	311	1066	1281	416	55
x		783	530	61	492	59	562	273	1064	19
	37									
k	$\xi 36$	18	19	20	21	22	23	24	25	26
w	$P1332$	702	775	884	1065	58	671	528	205	854
z		630	631	596	489	238	1031	1248	313	1070
y		702	701	736	843	1094	301	84	1019	262
x		630	557	448	267	1274	661	804	1127	478

k	27	28	29	30	31	32	33	34	35
w	783	604	209	714	355	932	717	250	611
z	1215	136	605	174	607	104	393	934	647
y	117	1196	727	1158	725	1228	939	398	685
x	549	728	1123	618	977	400	615	1082	721

Tab. Ic.

k		0	1	2	3	4	5	6	7	8	9
w		780	781	742	623	344	565	146	87	748	389
z		40	121	242	443	804	665	346	487	728	349
y		780	699	578	377	16	155	474	333	92	471
x		40	39	78	197	476	255	674	733	72	431
	41										
k	$\xi 20$	10	11	12	13	14	15	16	17	18	19
w	$P320$	450	531	652	33	394	255	756	77	318	759
z		370	371	332	213	754	155	556	497	338	799
y		450	449	488	607	66	665	264	323	482	21
x		370	289	168	787	426	565	64	743	502	61

k		0	1	2	3	4	5	6	7	8
w		560	561	520	395	102	75	580	343	428
z		42	127	254	465	242	355	538	259	260
y		560	475	348	137	360	247	64	343	342
x		42	41	82	207	500	527	22	259	174
	43									
k	$\xi 14$	9	10	11	12	13				
w	$P602$	555	164	543	54	237	**47**	1035	1036	991	854
z		219	94	403	376	279		46	139	278	509
y		383	508	199	226	323	$\xi 23$	1035	942	803	572
x		47	438	59	548	365	$P1081$	46	45	90	227

k		4	5	6	7	8	9	10	11	12	13
w		533	925	581	927	491	653	930	356	242	1048
z		924	626	1064	812	261	193	10	678	886	174
y		157	455	17	269	820	888	1071	403	195	907
x		548	156	500	154	590	428	151	725	839	33
	47										
k	$\xi 23$	14	15	16	17	18	19	20	21	22	
w	$P1081$	451	291	1005	224	777	755	664	435	1011	
z		865	38	499	293	915	1031	135	458	1057	
y		216	1043	582	788	166	50	946	623	24	
x		630	790	76	857	304	326	417	646	70	

TAB. ID.

k		0	1	2	3	4	5	6	7	8	
w	**73**	585	586	515	300	454	32	429	493	548	
z		72	217	434	138	130	41	447	529	620	
y	$\xi 9$	585	440	223	519	527	616	210	128	37	
x	$P657$	72	71	142	357	203	625	228	164	109	
k		0	1	2	3	4	5	6	7	8	9	10
w	**89**	891	892	805	542	906	566	776	128	701	779	846
z		88	265	530	971	785	324	292	139	723	823	934
y	$\xi 11$	891	714	449	8	194	655	687	840	256	156	45
x	$P979$	88	87	174	437	73	413	203	851	278	200	133
k		0	1	2	3	4	5	6			
w	**127**	763	764	639	262	270	159	699			
z		126	379	758	500	746	222	825			
y	$\xi 7$	763	510	131	389	143	667	64			
x	$P889$	126	125	250	627	619	730	190			
k		0	1	2	3	4	5	6	7		
w		3856	3857	3602	2835	1044	1317	1606	1927		
z		256	769	1538	2819	1012	1253	1478	1671		
y		3856	3343	2574	1293	3100	2859	2634	2441		
x		256	255	510	1277	3068	2795	2506	2185		
	257										
k	$\xi 16$	8	9	10	11	12	13	14	15		
w	$P4112$	2312	2825	3594	763	3068	3309	3534	3727		
z		1800	1801	1546	779	3100	3373	3662	3983		
y		2312	2311	2566	3333	1012	739	450	129		
x		1800	1287	518	3349	1044	803	578	385		

Tab. Ie.

k		0	1
w	3^2	16	5
z		14	7
y	$\xi 2$	4	11
x	$P 18$	2	13

k		0	1
w	3^3	34	5
z		32	7
y	$\xi 2$	22	47
x	$P 54$	20	49

k		0	1
w	3^4	34	113
z		32	115
y	$\xi 2$	130	47
x	$P 162$	128	49

k		0	1
w	3^5	34	113
z		356	277
y	$\xi 2$	130	209
x	$P 486$	452	373

k		0	1
w	3^6	34	599
z		1328	763
y	$\xi 2$	130	695
x	$P 1458$	1424	859

k		0	1	2	3
w	5^2	36	97	34	83
z		84	33	86	47
y	$\xi 4$	16	67	14	53
x	$P 100$	64	3	66	17

k		0	1	2	3
w	5^3	236	297	434	283
z		184	33	486	47
y	$\xi 4$	316	467	14	453
x	$P 500$	264	203	66	217

k		0	1	2	3
w	5^4	1236	297	1434	2283
z		184	1033	2486	1547
y	$\xi 4$	2316	1467	14	953
x	$P 2500$	1264	2203	1066	217

k		0	1	2
w	7^2	36	121	137
z		132	61	101
y	$\xi 3$	15	86	46
x	$P 147$	111	26	10

k		0	1	2
w	7^2	918	856	137
z		1014	649	542
y	$\xi 3$	15	380	487
x	$P 1029$	111	173	892

k		0	1	2	3	4
w	31^2	435	591	717	2643	2279
z		1735	2726	337	1883	2929
y	$\xi 5$	3070	2079	4468	2922	1876
x	$P 4805$	4370	4214	4088	2162	2526

k		0	1	2	3	4	5	6	7	8	9
w	11^2	430	981	532	833	104	65	636	7	718	1139
z		670	31	612	113	534	1035	376	1137	228	709
y	$\xi 10$	540	1179	598	1097	676	175	834	73	982	501
x	$P 1210$	780	229	678	377	1106	1145	574	1203	492	71

k		0	1	2	3	4	5	6	7	8	9	10	11
w	13^2	300	769	1382	1659	640	305	1650	271	1088	1461	10	1619
z		636	1285	74	447	1024	605	1314	1783	368	645	1654	1319
y	$\xi 12$	1392	743	1954	1581	1004	1423	714	245	1660	1383	374	709
x	$P 2028$	1728	1259	646	369	1388	1723	378	1757	940	567	2018	409

k		0	1	2	3	4	5	6	7
w	17^2	936	2161	2282	1283	356	117	2206	1063
z		1512	1273	1050	2219	2092	1005	1126	127
y	$\xi 8$	800	1039	1262	93	220	1307	1186	2185
x	$P 2312$	1376	151	30	1029	1956	2195	106	1249

TAB. II.

p	$w \not> 100$	$z \not> 100$	$y \not> 100$	$x \not> 100$	$x > 100$, but $\not> 250$
43	54,75	42,94	64	22,41,42,47, 59,82	174,207
47		10,38,46	17,24,50	33,45,46,70, 76,90	151,154,156,227
53	36,38,67	35,52	5,21,44,79	27,51,52	102,112,144,188
59	64,97	24,58	13,65,76,78, 88	30,57,58,79	114,167,243
61	43,67	17,60	91	31,59,60	118,188,232
67		66,73	24,100	34,65,66,80	130,152,173,181, 208
71	43	11,31,70,92	36,45,64	69,70,93	106,138
73	32	41,72	37	71,72	109,142,164,203, 228
79		12,57,74,78	18,40	77,78,93	118,154,182,188
83	35,90	60,82		29,42,81,82	162
89		88	8,45	73,87,88	133,174,200,203
97	11	31,96	49	95,96	145,170,190,197, 205
101		100	68	51,58,99,100	167,198
103	72	27	52,91	20	101,102,154,202
107	36			18,54,66	105,106,112,210
109		77		55	107,108,152,214, 241
113			57	64	111,112,162,169, 173,222
127			64		125,126,190,250
131	23		80	66	129,130,224,231
137		50,61	69,78		135,136,205,240
139	75	60,63		70	137,138,163
149	41			30,65,75	147,148,239
151			76	23	149,150,226
157				14,79	155,156
163		92		43,50,82	161,162,170
167			84		165,166,250
173		63		71,87	171,172
179	100	26,29	7	90	177,178
181	51,77			91	179,180
191			43,96		116,189,190,226
193			97	69	191,192
197	26	54,77	57	99	195,196
199	63		100	12	197,198,220
49	36	61	15,46,86	10,26	111
81	34	32	47	49	128
121	7,65	31	73	71	229
125		33,47	14	66	203,217

TAB. III.

p	$w \not\equiv 100$	p	$z \not\equiv 100$	p	$y \not\equiv 100$	p	$x \not\equiv 100$	p	$x > 100$, but $\not\equiv 250$
229	60	227	79	223	32	211	18,50	211	106,163
233	64	241	16	229	26	277	73	227	114,167
271	25	263	76	263	31	367	75	229	115,123,177
281	81	277	76	271	9,22,49	379	48	233	214
317	29	307	17	281	49,64	409	30	239	194
349	22	313	36	307	27	419	9	241	113,249
359	47	317	96	367	14	421	23	251	126,241
401	35	353	21	379	84	487	14	263	168
421	95	401	94	383	6	563	50	269	135,166
431	72	409	41	421	83	607	23	277	139,145,218
457	27	457	21	479	63	683	8	283	142
479	92	463	53	571	82	857	50	293	147
487	84	499	78	587	35	887	93	307	154
491	69	521	9	661	26	929	98	317	159
503	9	547	13,67	733	29	997	15,48	331	128,166,203
509	27	587	43,56,62	769	100			347	174,218
523	100	607	62	773	43,73			349	175
541	73	739	57	853	95			353	182
577	27	859	31	863	68			367	238
739	80	911	14,44	883	43			373	187
773	24	937	48	941	56			379	174,190
839	56			983	31			389	195
881	84,90							397	199
997	40							419	210
								421	211
								431	239
1021	12	1201	27	1009	29	1021	56	433	127
1103	64	1217	51	1433	15	1061	34	443	222
1217	91	1259	52	1439	48	1399	24	461	231
1259	80	1429	99	1559	21	1483	31	467	234
1367	79	1447	33	1609	55	1733	11	491	246
1423	23	1487	29	1747	40	1777	32	499	145,250
1493	28	1867	51	1879	20	1933	36	509	255
1637	14	1949	100	2137	12	2267	60	521	211
1741	37	1973	85	2339	29	2357	65	523	142,242
1877	40	2017	29,77	2503	11	2459	25	587	219
2003	22	2111	70	2687	84	2971	84	647	111
2089	64	2441	51	2999	28	4691	17	659	111
2243	68	2777	83	3089	98	5233	24	677	241
2473	36	2843	78,94	3413	10			701	154,210
2711	75	2917	97					709	207
2713	24	3083	85			5279	19	733	218
3169	90	3319	35	8963	48	5591	53	761	225
3329	38	4723	63	9377	60	6011	72	811	128
3517	100	4799	22			6689	25	821	123
3593	43	4861	99			7177	22	839	218
4679	15	4969	52			7541	20	881	117
5237	92	5569	72			8353	66	937	216
5927	48	6047	45					983	120
6733	31	6899	18						
6961	98	7109	24						
7517	66	9437	40						
8179	13								
8737	17								
8753	92								
9341	75								
9679	71								
9949	60								

Add $x=(p-2)$ & $(p-1)$ always

Add $x=(p-2)$ & $(p-1)$ always

Add $x=(p-2)$ & $(p-1)$ always

Factorisation of W, Z, Y, X.

TAB. IVA.

q	$W=2^w-w$, $[w=q]$	$Z=2^z+z$, $[z=q]$	$Y=y.2^y-1$, $[y=q]$	$X=x.2^x+1$, $[x=q]$
1	1;	3;	1;	3;
2	2;1;	2;3;	7;	9;
3	5;	11;	23;	25;
4	4;3;	4;5;	9;7;	5;13;
5	27;	37;	3.53;	7.23;
6	2;29;	2;5.7;	383;	5.7.11;
7	121;	27.5;	5.179;	3.13.23;
8	8;31;	8;3;11;	23.89;	3;683;
9	503;	521;	17.271;	11.419;
10	2;3.13.13;	2;11.47;	3.3413;	49.11.19;
11	3.7.97;	29.71;	9 2503;	13.1733;
12	4;1021;	4;13.79	23.2137;	13.19.199;
13	8179;	3.5.547;	5.19.19.59;	9.11833;
14	2;5.1637;	2;9.911;	625.367;	3.157.487;
15	7.4679;	32783;	343.1433;	17.29.997;
16	16;7|9||5.13;	16;17.241;	3;11.31||5;5.41;	17.61681;
17	3.5.8737;	7.61.307;	3.47.15803;	25.19.4691;
18	2;1310063;	2;19.6899;	79.59729;	11.19.107.211;
19	524269;	3.7.24967;	13 29.26423;	3.17.37.5279;
20	4;262139;	4;3.87383;	1879.11161;	27.103.7541;
21	2097131;	13.353.457;	13.41.53.1559;	23.1914791;
22	2;3.349.2003;	2;19.23.4799;	27.271.12611;	13.23.43.7177;
23	9.5.131.1423;	31.270601;	3.7.9187523;	5.151.421.607;
24	8;773.2713;	8;5.59.7109;	47.67.127867;	5.11.1399.5233;
25	271.123817;	9.3728273;	7.119837257;	3.17.2459.6689;
26	2;197.170327;	2;3.5.179.12497;	229.661.11527;	3.5.49.2373919;
27	509;457.577;	5.103;7.31.1201;	5.307;2360833;	29.53;7.336823;
28	4;3.1493.14983;	4;23.29.100613;	3.19.2999.43969;	29.29.8937209;
29	3.317.504533;	179.1487.2017;	9.733.1009.2339;	31.83.6051013;
30	2;17.19.19. 87481;	2;31.17318417;	‡	17.17.31.59. 149.409;
31	6733.318949;	3.121.71.97.859;	263.983.257503;	9.7.61.1483. 11681;
32	32;7;73;262657;	32;3;3;3.19;3. 87211;	223.616318177;	3;1777. 25781083;
33	41.209510599?	125.1447.47491;	5. ‡	47. ‡
34	2;27.81.25. 157109;	2;	3.5.29. ‡	19.23.67.1061 18803;
35	3;83.401.344117;	53.3319.195329;	3.587.6829022397	37.
36	4;53.2473|| 25.49.107;	4;37.313. 1483453;	71.22153|| 5.7.44939;	37.37.1933. 934861;
37	5.7.1741. 2255501;	3.11.13.	73.	3.5
38	2.29.53.3329. 26861;	2;3.7.17.47. 8191129;	17.	9.
39	11.			41.
40	8;3.997.1877. 24481;	8;7.41. 47888137 1?	9.79.1747. 35407667;	41. ‡

TAB. IVB.

q	$W=2^w-w$, $[w=q]$	$Z=2^z+z$, $[z=q]$
41	9.149.	73.409.73651849;
42	2.	2.43.
43	5.61.71.3593.113051;	9.17.19.587.5154739;
44	4.	4.3.5.911.32184753 1?
45		11.6047.528953801?
46	2.3.	2.5.11.47.9437.1442299;
47	3.359.	125.
48	16.5927.29681434 19?	16.7.37.937.72490393;
49		3.11.17.
50	2.	2.9.137.
51	181.	37.1217.1867.2441.10973;
52	4.9.	4.17.19.53.1259.4969.10513;
53	3.7.	5.17.463.
54	2.5.43.	2.197.
55		3.19.
56	8.5.839.	8.3.13.587.
57	5.7.13.	79.739.
58	2.3.7.11.37.	2.59.
59	27.13.17.	7.
60	4.229.9949.1465999449?	4.23.61.83.139.17806935469?
61	29.	27.49.137.
62	2.	2.3.11.587.607.
63	5.11.199.	139.173.4723.
64	64;233.1103.2089|3;59.3033169;	64;5.107367629:536903681; L
65	3.121.	
66	2.7517.	2.5.13.67.

q	$Y=y.2^y-1$, $[y=q]$	$X=x.2^x+1$, $[x=q]$
41	3.19.	13.43.
42		23.43.83.
43	29.191.773.883.100043;	3.5.163.
44	7.53.	3.5.
45	71.89.	11.47.
46	3.49.	5.31.47.
47	81.5.	7.11.43.
48	11.1439.8963.95230081;	7.379.997. ‡
49	97.271.281.	81.
50	47.	3.163.211.563.857.1130809;
51		53.101.
52	3.103.	7.11.17.53.
53	3.25.	5591.
54	5.	29.107.
55	37.1609.	3.109.
56	5.941. ‡	9.1021. ‡
57	7.113.197.	5.13.59.
58	9.	31.59.101.
59	3.23.	37.43.61.
60	9377. ‡	61.2267. ‡
61	11.	3.19.37.41.
62	13.23.37. ‡	3. ‡
63	479. ‡	5. ‡
64	31;127;71122921||3;11;43;281.86171;	5;5:41;29:113;47392381:7416361;
65	9.7.11.59. ‡	67.149.2357. ‡
66	41. ‡	125.13.67.107.131.8353. 382030403? ‡

TAB. Va.

Small Factors of $X = x \cdot 2^x + 1$.

x	X	x	X	x	X
61	3.19.37.41.	95	29.97.	129	131.
62	3.	96	97.	130	67.131.
63	5.	97	3.5 13.	131	7.
64	25.29.41.113.	98	3.929.	132	7.
65	67.149.2357.	99	13.101.197.	133	3.29.89.
66	125.13.67.107.131.	100	101.	134	3.11.
67	9.	101	103.	135	23.137.269.
68	3.7.	102	53.103.	136	7.137.
69	7.71.193.	103	27.25.11.	137	5.29.139.
70	37.47.71.139.	104	3.5.	138	71.139.
71	121.73.173.	105	107.	139	9.19.277.
72	41.73.6011.	106	5.17.17.71.107.211.	140	3.13.
73	3.7.89.277.	107	23.109.	141	‡
74	27.	108	109.	142	73.283.523.
75	149.367.	109	3.73.	143	5.
76	47.	110	9.7.	144	5.53.
77	5.17.79.	111	343.113.647.659.	145	3.97.277.499.
78	41.79.	112	53.107.113.	146	9.5.13.
79	3.59.157.	113	241.	147	31.149.293.
80	3.67.	114	59.227.	148	149.
81	83.	115	3.7.229.	149	19.151.
82	43.83.163.	116	3.11.19.191.	150	151.
83	5.23.	117	25.881.	151	3.17.17.47.
84	5.2971.	118	61.79.	152	3.7.17.67.109.
85	9.	119	11.19.	153	7.19.
86	3.5.	120	11.983.	154	47.79.103.307.701.
87	19.89.173.	121	9.	155	11.17.157.
88	89.	122	3.	156	47.157.
89	7.	123	5.229.821.	157	27.5.49.11.
90	7.47.179.	124	5.	158	3.
91	3.181.	125	23.127.	159	317.
92	9.	126	5.127.251.	160	13.6397.
93	71.79.887.	127	3.433.		
94	7.	128	81.11.19.331.811.		

TAB. Vb.

Small Factors of $X=x \cdot 2^x+1$.

x	X	x	X	x	X	
161	17.163.	195	7.197.389.	228	73.229.	
162	11.83.113.163.	196	197.	229	9.121.	
163	3.5.13.139.211.	197	5.13.41.97.199.	230	3.11.	
164	9.25.73.	198	101.199.	231	131.233.461.	
165	167.	199	3.7.19.397.	232	61.233.	
166	25.17.167.269.331.	200	9.89.	233		†
167	13 59.101.227.	201	31.	234	467.	
168	13.41.263.	202	103.	235	3.	
169	3.113.	203	125.73.89.331.	236	27.7.29.	
170	3.97.163.	204	5.	237	5.7.239.	
171	173.	205	3.97.137.	238	11.239.367.	
172	173.	206	3.5.	239	149.241.431.	
173	343.67.113.	207	43.709.	240	137.241.	
174	7.43.89.347.379.	208	67.	241	3.7.109.251.677.	
175	9.349.	209	211.	242	3.17.523.	
176	3.	210	107.211.419.701.	243	5.59.	
177	5.29.37.179.229.	211	81.421.521.	244	5.11.	
178	7.13.23.179.	212	3.19.	245		†
179	181.	213	11.13.17.31.	246	5.491.	
180	181.	214	109.233.	247	9.	
181	3.11.67.	215	7.	248	3.	
182	27.79.353.	216	7.937.7873.	249	241.251.	
183	5.	217	3.625.	250	127.167.251.499.	
184	5.19.31.	218	9.277.347.733.839.	251		†
185	29.31.	219	587.	252		†
186	5.	220	7.199.	253	3.13.	
187	3.373.	221	223.	254	9.53.113.	
188	3.17.53.61.79.	222	13.113.223.443.	255	13.41.61.257.509.	
189	191.	223	3.5.	256	97.257.673.	
190	97.127.191.379.	224	3.5.131.	257	5.7.53.	
191	193.	225	227.761.	258	49.23.97.131.167.337.	
192	193.	226	5.11.151.191.227.	259	3.43.71.83.	
193	9.	227	47.229.	260	3.23.	
194	3.7.239.1579.					

CURVES OF CONSTANT TORSION.

By *W. H. Salmon.*

The equations

$$\left.\begin{aligned} x &= \tau \int \frac{BC' - B'C}{A^2 + B^2 + C^2}\, dt \\ y &= \tau \int \frac{CA' - C'A}{A^2 + B^2 + C^2}\, dt \\ z &= \tau \int \frac{AB' - A'B}{A^2 + B^2 + C^2}\, dt \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (1),$$

where A, B, C are functions of t, and A', B', C' their differential coefficients with respect to t, represent a curve of constant torsion τ.

It has been shown by M. Lyon* that if A, B, C be so chosen that the resulting curves are cubic or quartic, then these curves are imaginary. The aim of this paper is to obtain (i) an infinite series of curves of constant torsion traced on algebraic cylinders, the circular helix being the simplest member of the series; (ii) an infinite series of algebraic curves of constant torsion. It will be seen that the second series is a particular case of the first series when the period is zero, and that they include as special cases curves already discovered by M. Lyon and M. Fabry. If

$$\left.\begin{aligned} A &= a \cos (bt + c) \\ B &= a \sin (bt + c) \cos (et + f) \\ C &= a \sin (bt + c) \sin (et + f) \end{aligned}\right\} \ldots\ldots\ldots\ldots (2)$$

the equations (1) for x, y and z are integrable at once, and we shall see that under certain conditions they will represent an infinite series of curves traced on algebraic cylinders.

A very special case occurs when $b = 0$. Then

$$A = a \cos c, \qquad A' = 0,$$

$$B = a \sin c . \cos (et + f), \quad B' = -ae \sin c . \sin (et + f),$$

$$C = a \sin c . \sin (et + f), \quad C' = ae \sin c . \cos (et + f).$$

* 'Sur les courbes à torsion constante,' *Annales de l'Enseignement supérieur de Grenoble*, t. ii., p. 353 (1890).

Hence, from (1),

$$\left.\begin{aligned} x &= \tau e \sin^2 c . t \\ y &= -\tau \sin c . \cos c . \sin (et+f) \\ z &= \tau \sin c . \cos c . \cos (et+f) \end{aligned}\right\} \ldots\ldots\ldots\ldots (3).$$

This curve is the *Circular Helix.*

Returning to the general formulæ (2), and assuming now that b is finite, we may without loss of generality omit a, which cancels out in the numerators and denominators of x, y and z, c which increases or diminishes t by a constant, and b. Hence instead of (2) we may equivalently write

$$\left.\begin{aligned} &A = \cos t, && A' = -\sin t \\ &B = \sin t . \cos (et+f), && B' = \cos t . \cos (et+f) - e \sin t . \sin (et+f) \\ &C = \sin t . \sin (et+f), && C' = \cos t . \sin (et+f) + e \sin t . \cos (et+f) \end{aligned}\right\}$$

$\ldots\ldots\ldots (4).$

Hence

$$\left.\begin{aligned} x &= \tau \int e \sin^2 t dt = \tfrac{1}{2}\tau e \{t - \tfrac{1}{2} \sin 2t\} \\ y &= \tau \int \{-\sin (et+f) - \tfrac{1}{2} e \sin 2t . \cos (et+f)\} dt \\ &= \tau \left\{ \frac{1}{e} \cos (et+f) - \frac{e}{4(e-2)} \cos [(e-2)t + f] \right. \\ &\qquad \left. + \frac{e}{4(e+2)} \cos [(e+2)t + f] \right\} \\ z &= \tau \int \{\cos (et+f) - \tfrac{1}{2} e \sin 2t \sin (et+f)\} dt \\ &= \tau \left\{ \frac{1}{e} \sin (et+f) - \frac{e}{4(e-2)} \sin [(e-2)t + f] \right. \\ &\qquad \left. + \frac{e}{4(e+2)} \sin [(e+2)t + f] \right\} \end{aligned}\right\}$$

$\ldots\ldots\ldots (5).$

These equations represent, for different values of e and f, an infinite series of curves of constant torsion. The most interesting case is when $f = 0$ and e is any integer n except ± 2. The equations (5) then become

$$\left.\begin{aligned} x &= \tfrac{1}{2}\tau n\{t - \tfrac{1}{2}\sin 2t\} \\ y &= \tau\left\{\frac{1}{n}\cos nt - \frac{n}{4(n-2)}\cos(n-2)t + \frac{n}{4(n+2)}\cos(n+2)t\right\} \\ z &= \tau\left\{\frac{1}{n}\sin nt - \frac{n}{4(n-2)}\sin(n-2)t + \frac{n}{4(n+2)}\sin(n+2)t\right\} \end{aligned}\right\} \quad \ldots\ldots\ldots(6),$$

which represent an infinite series of curves of constant torsion traced on rational* algebraic cylinders. Since x and y are odd functions and z an even function of n, we can always take n a *positive* integer.

As stated above an exception occurs when $n = 2$, in which case the above integrations do not apply. We have then

$$\left.\begin{aligned} x &= \tau\int 2\sin^2 t\,dt = \tau\{t - \tfrac{1}{2}\sin 2t\} \\ y &= \tau\int\{-\sin 2t - \sin 2t\cos 2t\}\,dt = \tau\{\tfrac{1}{2}\cos 2t + \tfrac{1}{8}\cos 4t\} \\ z &= \tau\int\{\cos 2t - \sin^2 2t\}\,dt = \tau\{\tfrac{1}{2}\sin 2t + \tfrac{1}{8}\sin 4t - \tfrac{1}{2}t\} \end{aligned}\right\} \ldots(7).$$

This curve is also a curve traced on an algebraic cylinder, as can be seen at once by referring the curve to the lines

$$x + 2z = 0,\quad y = 0;\quad x = 0,\quad z = 0;\quad 2x - z = 0,\quad y = 0$$

as new axes.†

The next section will shew that the series (6) is only a particular case of a more general series.

If we take

$$\left.\begin{aligned} A &= a_1\cos\alpha_1 t + a_2\cos\alpha_2 t + a_3\cos\alpha_3 t \\ B &= a_1\sin\alpha_1 t + a_2\sin\alpha_2 t + a_3\sin\alpha_3 t \\ C &= b_0 + b_1\cos\delta t \end{aligned}\right\} \quad \ldots\ldots\ldots(8),$$

---

* For y and z can be expressed as rational functions of $\tan\frac{1}{2}t$.

† It can easily be shewn, by a change of axes and parameter, that this particular curve of the series is identical with the curve

$$x - x_0 = -\tfrac{1}{2}\tau\sqrt{5}\left[\frac{t(t^2+\frac{7}{5})}{(t^2+1)^2} - \tan^{-1}t\right]$$

$$y - y_0 = \frac{\tau}{\sqrt{\{5(1+a^2)\}}}\cdot\frac{2t - \sqrt{5}a}{(t^2+1)^2}$$

$$z - z_0 = \frac{\tau}{\sqrt{\{5(1+a^2)\}}}\cdot\frac{2at + \sqrt{5}}{(t^2+1)^2},$$

discovered by M. Lyon in 'Sur les courbes à torsion constante,' *Annales de l'Enseignement supérieur de Grenoble*, t. ii., p. 353 (1890). It should be noted that the value of a does not affect the shape of the curve, but merely determines its orientation. No particularity therefore is introduced by taking $a = 0$.

where $\delta = \alpha_2 - \alpha_1 = \alpha_3 - \alpha_2$, then

$$A^2 + B^2 + C^2 = a_1^2 + a_3^2 + a_3^2 + 2(a_1a_2 + a_2a_3)\cos\delta t + 2a_1a_3\cos 2\delta t + b_1^2\cos^2\delta t + 2b_0b_1\cos\delta t + b_0^2.$$

If in addition we so choose b_0 and b_1 that the coefficients of $\cos dt$ and of $\cos^2 dt$ vanish, then $A^2 + B^2 + C^2$ will be a constant. The conditions for this are

$$b_1^2 + 4a_1a_3 = 0, \quad b_0b_1 + a_1a_2 + a_2a_3 = 0,$$

i.e. $$b_1 = \pm 2\sqrt{(-a_1a_3)}, \quad b_0 = -a_2(a_1 + a_3)/2\sqrt{(-a_1a_3)} \ldots (9).$$

Denoting $A^2 + B^2 + C^2$ which is now constant by k^2, we have

$$x = \frac{\tau}{k^2}\int (BC' - B'C)\,dt$$

$$= \frac{\tau}{k^2}\left[-\tfrac{1}{2}b_1\delta\left\{a_1\left(\frac{\sin(\delta-\alpha_1)t}{\delta-\alpha_1} - \frac{\sin(\delta+\alpha_1)t}{\delta+\alpha_1}\right)\right.\right.$$

$$\left. + a_2\left(\frac{\sin(\delta-\alpha_2)t}{\delta-\alpha_2} - \frac{\sin(\delta+\alpha_2)t}{\delta+\alpha_2}\right) + a_3\left(\frac{\sin(\delta-\alpha_3)t}{\delta-\alpha_3} - \frac{\sin(\delta+\alpha_3)t}{\delta+\alpha_3}\right)\right\}$$

$$-\tfrac{1}{2}b_1\left\{a_1\alpha_1\left(\frac{\sin(\delta-\alpha_1)t}{\delta-\alpha_1} + \frac{\sin(\delta+\alpha_1)t}{\delta+\alpha_1}\right) + a_2\alpha_2\left(\frac{\sin(\delta-\alpha_2)t}{\delta-\alpha_2} + \frac{\sin(\delta+\alpha_2)t}{\delta+\alpha_2}\right)\right.$$

$$\left. + a_3\alpha_3\left(\frac{\sin(\delta-\alpha_3)t}{(\delta-\alpha_3)} + \frac{\sin(\delta+\alpha_3)t}{\delta+\alpha_3}\right)\right\}$$

$$\left. - b_0\left\{a_1\sin\alpha_1 t + a_2\sin\alpha_2 t + a_3\sin\alpha_3 t\right\}\right]$$

and

$$y = \frac{\tau}{k^2}\int (CA' - C'A)\,dt$$

$$= \frac{\tau}{k^2}\left[-\tfrac{1}{2}b_1\delta\left\{a_1\left(\frac{\cos(\delta+\alpha_1)t}{\delta+\alpha_1} + \frac{\cos(\delta-\alpha_1)t}{\delta-\alpha_1}\right)\right.\right.$$

$$\left. + a_2\left(\frac{\cos(\delta+\alpha_2)t}{\delta+\alpha_2} + \frac{\cos(\delta-\alpha_2)t}{\delta-\alpha_2}\right) + a_3\left(\frac{\cos(\delta+\alpha_3)t}{\delta+\alpha_3} + \frac{\cos(\delta-\alpha_3)t}{\delta-\alpha_3}\right)\right\}$$

$$+\tfrac{1}{2}b_1\left\{a_1\alpha_1\left(\frac{\cos(\delta+\alpha_1)t}{\delta+\alpha_1} + \frac{\cos(\delta-\alpha_1)t}{\delta-\alpha_1}\right) + a_2\alpha_2\left(\frac{\cos(\delta+\alpha_2)t}{\delta+\alpha_2} + \frac{\cos(\delta-\alpha_2)t}{\delta-\alpha_2}\right)\right.$$

$$\left. + a_3\alpha_3\left(\frac{\cos(\delta+\alpha_3)t}{\delta+\alpha_3} - \frac{\cos(\delta-\alpha_3)t}{\delta-\alpha_3}\right)\right\}$$

$$\left. + b_0\left\{a_1\cos\alpha_1 t + a_2\cos\alpha_2 t + a_3\cos\alpha_3 t\right\}\right] \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

where b_0, b_1 have the values given in (9), and provided that δ is not equal to $\pm\alpha_1$, $\pm\alpha_2$ or $\pm\alpha_3$.

With the same conditions

$$z=\frac{\tau}{k^2}\int(AB'-A'B)\,dt$$

$$=\frac{\tau}{k^2}\left[\left\{a_1{}^2\alpha_1+a_2{}^2\alpha_2+a_3{}^2\alpha_3\right\}t+\left\{a_1a_2(\alpha_1+\alpha_2)+a_2a_3(\alpha_2+\alpha_3)\right\}\frac{\sin\delta t}{\delta}\right.$$

$$\left.+a_1a_3(\alpha_1+\alpha_3)\,\frac{\sin 2\delta t}{2\delta}\right]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

If α_1, α_2, α_3 are finite integers, and a_1 and a_3 are of opposite signs; if $\delta=\alpha_3-\alpha_2=\alpha_2-\alpha_1$ and δ is not equal to $\pm\alpha_1$, $\pm\alpha_2$ or $\pm\alpha_3$; if in addition b_0 and b_1 have the values assigned in (9), then (10) and (11) represent an infinite series of real curves of constant torsion traced on algebraic cylinders.

We may note particularly the special case when $a_1=1$, $a_2=0$, $a_3=-1$, $\alpha_1=1-n$, $\alpha_2=-n$, $\alpha_3=-1-n$, n being an integer, giving

$$A=2\sin t\sin nt,\quad B=2\sin t\cos nt,\quad C=2\cos t,$$

which is equivalent to equations (4) if $d=n$ and $e=0$. This case is therefore identical with the infinite series of twisted curves of constant torsion given by (6).

Finally, if in addition to the conditions stated immediately after (11) we have

$$a_1{}^2\alpha_1+a_2{}^2\alpha_2+a_3{}^2\alpha_3=0\ \ldots\ldots\ldots\ldots\ldots(12),$$

then equations (10) and (11) represent *an infinite series of algebraic curves of constant torsion.*

In particular we may note that if $a_1=\sqrt{\lambda}$, $a_2=0$, $a_3=-\sqrt{\mu}$, $\alpha_1=\mu$, $\alpha_2=\frac{1}{2}(\mu-\lambda)$, $\alpha_3=\lambda$, we have as a special case of the above the formulæ due to M. Fabry* for a series of algebraic curves of constant torsion.

It is obvious that the above method may be extended to any odd number of finite integers α_1, α_2, α_3, ..., α_{2n+1} in arithmetical progression, and that we may thus obtain an infinite number of algebraic curves of constant torsion. We shall consider this general case in the next section. If

$$\left.\begin{aligned}A&=a_1\cos\alpha_1t+a_2\cos\alpha_2t+\ldots\quad+a_{2n+1}\cos\alpha_{2n+1}t\\B&=a_1\sin\alpha_1t+a_2\sin\alpha_2t+\ldots\quad+a_{2n+1}\sin\alpha_{2n+1}t\\C&=b_0+b_1\cos\delta t+b_2\cos 2\delta t+\ldots+b_n\cos n\delta t\end{aligned}\right\}\ldots(13),$$

* 'Sur les courbes à torsion constante,' *Annales de l'Ecole Normale supérieure*, 3e série, t. ix., p. 177 (1892).

where $\alpha_1, \alpha_2, \ldots, \alpha_{2n+1}$ are an odd number of integers in arithmetical progression of common difference δ, and δ is not equal to $\pm\alpha_1, \pm\alpha_2, \ldots$, or $\pm\alpha_{2n+1}$, then we have $2n$ equations connecting the coefficients $b_0, b_1, \ldots, b_n$, and the coefficients $a_1, a_2, \ldots, a_{2n-1}$, in order that $A^2+B^2+C^2$ may be independent of t. For

$$A^2+B^2+C^2=\Sigma_{r=1}^{r=2n+1}a_r^2+2\cos\delta t\,\Sigma_{r=1}^{r=2n}a_ra_{r+1}$$
$$+2\cos 2\delta t\,\Sigma_{r=1}^{r=2n-1}a_ra_{r+2}+\ldots+2\cos 2n\delta t\,.\,a_1a_{2n+1}$$
$$+b_0^2+\Sigma_{r=1}^{r=n}b_r^2\cos^2 r\delta t+2b_0\Sigma_{r=1}^{r=n}b_r\cos r\delta t$$
$$+b_1\Sigma_{r=2}^{r=n}b_r\{\cos(r+1)\,\delta t+\cos(r-1)\,\delta t\}$$
$$+b_2\Sigma_{r=3}^{r=n}b_r\{\cos(r+2)\,\delta t+\cos(r-2)\,\delta t\}+\ldots$$
$$+b_{n-2}\Sigma_{r=n-1}^{r=n}b_r\{\cos(r+n-2)\,\delta t+\cos(r-n+2)\,\delta t\}$$
$$+b_{n-1}b_n\{\cos(2n-1)\,\delta t+\cos\delta t\}$$
$$=\Sigma_{r=1}^{r=2n+1}a_r^2+2\Sigma_{s=1}^{s=2n}\left[\cos s\delta t\,\Sigma_{r=1}^{r=2n-s+1}a_ra_{r+s}\right]$$
$$+b_0^2+\tfrac{1}{2}\Sigma_{r=1}^{r=n}b_r^2(1+\cos 2r\delta t)+2b_0\Sigma_{r=1}^{r=n}b_r\cos r\delta t$$
$$+\Sigma_{s=1}^{s=n-1}\left[b_s\Sigma_{r=s-1}^{r=n}b_r\{\cos(r+s)\,\delta t+\cos(r-s)\,\delta t\}\right]\ldots\ldots(14).$$

Equating coefficients of $\cos\delta t, \cos 2\delta t, \ldots, \cos 2n\delta t$ to zero, we have $2n$ equations obtained by giving k the successive values 1 to $2n$ in the following typical equation*:—

$$2\Sigma_{r=1}^{r=2n-k+1}a_ra_{r+k}+2b_0b_k+\tfrac{1}{2}b_{\frac{1}{2}k}^2\cos^2\tfrac{1}{2}k\pi+\Sigma_{r=1}^{r=n-k}b_rb_{r+k}+\tfrac{1}{2}\Sigma'b_rb_s=0$$
$$\ldots\ldots\ldots(15),$$

the summation Σ' including all unequal integral values of r and s from 1 to n which make $r+s=k$. If the coefficients satisfy equation (15), and if in addition

$$a_1^2\alpha_1+a_2^2\alpha_2+\ldots+a_{2n+1}^2\alpha_{2n+1}=0\ldots\ldots\ldots\ldots(16),$$

the curve obtained from (13) is a rational algebraic curve of constant torsion, for x, y, z can be expressed rationally in terms of $\tan\frac{1}{2}t$.

Substituting the values of A, B, C in (13) in formula (1) and integrating, we obtain the following infinite series of rational algebraic curves of constant torsion τ, viz.

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* It is of course obvious that if $k>n$ there are no terms of the type $2b_0b_k$. The last term of the equation is $\frac{1}{2}\Sigma'b_rb_s$, the symbol Σ' implying that b_rb_s is distinct from b_sb_r.

$$x=\frac{\tau}{k^2}\left[-b_0\Sigma_{r=1}^{r=2n+1}a_r\sin\alpha_r t-\tfrac{1}{2}\Sigma'\left\{\frac{a_r b_s(\alpha_r+s\delta)}{\alpha_r-s\delta}\sin(\alpha_r-s\delta)t+\frac{a_r b_s(\alpha_r-s\delta)}{\alpha_r+s\delta}\sin(\alpha_r+s\delta)t\right\}\right]$$

$$y=\frac{\tau}{k^2}\left[b_0\Sigma_{r=1}^{r=2n+1}a_r\cos\alpha_r t+\tfrac{1}{2}\Sigma'\left\{\frac{a_r b_s(\alpha_r-s\delta)}{\alpha_r+s\delta}\cos(\alpha_r+s\delta)t+\frac{a_r b_s(\alpha_r+s\delta)}{\alpha_r-s\delta}\cos(\alpha_r-s\delta)t\right\}\right]$$

$$z=\frac{\tau}{k^2}\Sigma''\left[\frac{a_r a_s\alpha_s}{\alpha_r-\alpha_s}\sin(\alpha_r-\alpha_s)t\right] \qquad \text{.....(17),}$$

where the sign of summation Σ' includes all combinations of all integral values of r from 1 to $2n+1$ and of s from 1 to n, and Σ'' all combinations of all unequal integral values of r and s from 1 to $2n+1$; where

$$k^2=A^2+B^2+C^2=\Sigma_{r=1}^{r=2n+1}a_r^2+b_0^2+\tfrac{1}{2}\Sigma_{r=1}^{r=n}b_r^2,$$

and $\alpha_1, \alpha_2, \ldots, \alpha_{2n+1}$ are any odd number of unequal integers in arithmetical progression of common difference δ which is not equal to nor is any submultiple greater than or equal to $1/n$ of $\pm\alpha_1, \pm\alpha_2, \ldots,$ or $\pm\alpha_{2n+1}$; and where $a_1, a_2, \ldots, a_{2n+1}$, $b_0, b_1, \ldots, b_n, \alpha_1, \alpha_2, \ldots, \alpha_n$ have any values subject to the conditions expressed in (15) and (16).

The $2n+1$ equations (15) and (16) connecting the $3n+4$ quantities $\alpha_1, \delta, a_1, \ldots, a_{2n+1}, b_0, \ldots, b_n$ have an infinite though restricted number of real solutions. For example, the values $a_1=1$, $a_2=\sqrt{\{32+8\sqrt{10}\}}$, $a_3=2+\sqrt{10}$, $a_4=0$, $a_5=-1$, $b_0=-(4+\sqrt{10})$, $b_1=\sqrt{\{32+8\sqrt{10}\}}$, $b_2=2$, $\alpha_1=-4$, $\delta=3$ satisfy these equations and lead on integration to a real curve of constant torsion. Instances might be multiplied.

It has long been known* that all real algebraic curves of constant torsion must be closed. The above series (17) of closed real algebraic curves of constant torsion is seen to be a particular case of periodic transcendental curves of constant torsion in which the periodic length is zero.

* G. Koenigs, 'Sur la forme des courbes à torsion constante,' *Annales de la Faculté des Sciences de Toulouse*, t. i., p. E 1 (1887).

THE CENTRE OF GRAVITY IN DARBOUX CO-ORDINATES.

By *T. C. Lewis, M.A.*

1. It has been proved* that any point whose co-ordinates are $x_1, x_2, \ldots, x_{n+2}$ is the mean centre of the vertices of reference for a system of multiples

$$x_1/\rho_1,\ x_2/\rho_2,\ \ldots,\ x_{n+2}/\rho_{n+2}.$$

Hence masses $m', m'', \ldots$, at points $(x_1', x_2', \ldots)$, $(x_1'', x_2'', \ldots)$, &c., are equivalent to masses

$$-(m'x_1'+m''x_1''+\ldots)/2\rho_1,\ -(m'x_2'+m''x_2''+\ldots)/2\rho_2, \text{ &c.},$$

at the vertices.

Let $(\bar{x}_1, \bar{x}_2, \ldots)$ be the centre of gravity of the given masses, which are therefore equivalent to

$$-\Sigma m'\frac{\bar{x}_1}{2\rho_1},\ -\Sigma m'\frac{\bar{x}_2}{2\rho_2}, \text{ &c.},$$

at the respective vertices.

Masses proportional to $1/\rho_1^2$, $1/\rho_2^2$, &c., at the vertices may be added without affecting the result.

Therefore to determine the co-ordinates of the centre of gravity we have

$$\bar{x}_1=\frac{m'x_1'+m''x_1''+\ldots}{m'+m''+\ldots}-\frac{h^2}{\rho_1},$$

$$\bar{x}_2=\frac{m'x_2'+m''x_2''+\ldots}{m'+m''+\ldots}-\frac{h^2}{\rho_2},$$

$$\text{&c.} = \text{&c.}$$

Here $\Sigma\dfrac{\bar{x}_k}{\rho_k}=-2$. We must also have $\Sigma\bar{x}_k^2=0$.

Hence
$$h^2=\frac{m'm''l_{12}^2+\text{similar terms}}{(m'+m''+\ldots)^2},$$

where l_{12} is the length of the line joining the particles m', m''. Thus in general

$$x_k=\frac{\Sigma m'x_k'}{\Sigma m'}-\frac{\Sigma m'm''l_{12}^2}{(\Sigma m')^2\rho_k} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

$$=\frac{\Sigma m'x_k'}{\Sigma m'}+\frac{\rho^2}{\rho_k},$$

* *Messenger of Mathematics*, vol. xlv., p. 85.

where the centre of gravity is the centre and ρ is the radius of the circle (&c.)

$$\Sigma(m'x_1')\,x_1+\Sigma(m'x_2')\,x_2+\ldots=0.$$

From this it is seen that if the masses are all of the same sign the radius of the circle is imaginary.

2. If we multiply $\bar{x}_k$ by x_k, and take the sum of the products for all values of k; and if d', d'', ..., and $\bar{d}$ are the distances of any point $(x_1, x_2, \ldots)$ from the particles m', m'', ..., and from their centre of gravity respectively, then

$$\bar{d}^2=\frac{\Sigma m'd'^2}{\Sigma m'}-\frac{\Sigma m'm''l''^2_{12}}{(\Sigma m')^2} \quad \ldots\ldots\ldots\ldots\ldots(2),$$

and therefore if the point $(x_1, x_2, \ldots)$ coincide with the centre of gravity

$$\Sigma m'.\Sigma m'd'^2=\Sigma m'm''l''^2_{12} \quad \ldots\ldots\ldots\ldots\ldots(3).$$

If there are n_1 particles, of equal masses, we obtain the formula

$$n_1\left(\bar{d}^2+\frac{\Sigma l^2}{n_1^2}\right)=\Sigma d'^2,$$

generalising the result of the case for which $n_1=2$. These propositions may easily be proved by Cartesian co-ordinates.

3. If instead of the $n+2$ co-ordinates of a point in n-space we employ the $n+1$ co-ordinates $\xi_1, \xi_2, \ldots, \xi_{n+1}$, where ξ_k is proportional to $\rho_k x_k-\rho_{n+2}x_{n+2}$, we obtain in place of (1) the formula

$$\bar{\xi}_k=\frac{\Sigma m'\xi'_k}{\Sigma m'} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

as it must be, since ξ_k is proportional to the perpendicular distance of the point from the $(n+1)$-dimension base opposite to the k^{th} vertex of the figure of reference.

4. If the particles m', m'', ..., be at the vertices A, B, ..., we obtain directly, by the method of paragraph 1, or we may deduce from (1), the co-ordinates of the centre of gravity in the form

$$\bar{x}_1=-2\frac{m'}{\Sigma m'}\rho_1+\frac{m'^2\rho_1^2+m''^2\rho_2^2+\ldots}{(\Sigma m')^2\rho_1}$$

$$=-2\frac{m'}{\Sigma m'}\rho_1+\frac{\rho^2}{\rho_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

where ρ is the radius of the circle, &c.,

$$m'\rho_1 x_1 + m''\rho_2 x_2 + \ldots = 0,$$

whose centre is at the centre of gravity.

Thus the co-ordinates of the centre of gravity, or the mean centre for a given system of multiples at the vertices, are determined in a simple and easy way.

For example, in two dimensions we have:

(i) For the mean centre of equal particles at the four vertices

$$x_k = -\tfrac{1}{2}\rho_k + \frac{\rho^2}{\rho_k},$$

where ρ is the radius of the nine-point circle.

(ii) For the system of multiples 1, 1, 1, 0,

$$x_k = -\tfrac{2}{3}\rho_k + \frac{\rho^2}{\rho_k} \quad (k = 1,\ 2 \text{ or } 3),$$

and $$\rho_4 x_4 = \rho^2,$$

where ρ is the radius of the 24-point circle,* *i.e.*

$$\rho^2 = \tfrac{1}{18}(a^2 + b^2 + c^2).$$

(iii) For the system a^2, b^2, c^2, 0

$$x_1 = -\frac{2a^2}{a^2 + b^2 + c^2}\rho_1 + \frac{\rho^2}{\rho_1},$$

where ρ is the radius of the circle

$$a^2\rho_1 x_1 + b^2\rho_2 x_2 + c^2\rho_3 x_3 = 0,$$

that is $$\rho^2 = 4k^2(R^2 - \rho_4^2),$$

where $$k = 2\Delta/(a^2 + b^2 + c^2).$$

The centre is the isogonal conjugate of the centroid of the triangle ABC, its distances from the sides being ka, kb, kc.† Similarly for x_2 and x_3, while $\rho_4 x_4 = \rho^2$.

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* *Messenger of Mathematics*, vol. xlv., p. 87.

† *Ibid*, p. 89.

FOURIER SERIES FOR THE SQUARES OF HERMITE'S EIGHTEEN DOUBLY PERIODIC THETA QUOTIENTS.

By *E. T. Bell.*

Having had occasion in the theory of numbers* to make repeated use of the series described in the title, I have thought it may be well, as only six appear in the standard treatises, to preserve all, together with their simple derivations and a few elementary, though arithmetically important, properties of the quotients. The notation

$$H,\ \Theta,\ H_1,\ \Theta_1 \equiv H(z),\ \Theta(z),\ H_1(z),\ \Theta_1(z)$$

respectively, where $z \equiv \dfrac{2Kx}{\pi}$ is that of Hermite,† who also writes θ, θ_1, η for $\Theta(0)$, $\Theta_1(0)$, $H_1(0)$ respectively. For convenience, the eighteen quotients have been expressed in terms of θ, θ_1, η and three functions u, v, w, whose squares were calculated by Jacobi in sections 41, 42 of the *Fundamenta Nova*. There is a great gain in symmetry when we consider the complete set of eighteen quotients, rather than a special three, as is usually done.

§ 1. *Definition of u, v, w. Constants.*

$$u \equiv \eta^{-1} \operatorname{sn} z = \eta\theta_1 \frac{H}{\Theta}, \quad \theta = \sqrt{\left(\frac{2k'K}{\pi}\right)},$$

$$v \equiv \eta^{-1} \operatorname{cn} z = \eta\theta \frac{H_1}{\Theta}, \quad \theta_1 = \sqrt{\left(\frac{2K}{\pi}\right)},$$

$$w \equiv \theta_1^{-2} \operatorname{dn} z = \theta\theta_1 \frac{\Theta_1}{\Theta}, \quad \eta = \sqrt{\left(\frac{2kK}{\pi}\right)}.$$

§ 2. *Hermite's quotients. Identities.*

We will denote each of Hermite's functions, (1) to (18), by its corresponding number, thus $(1) \equiv u$, $(1)^2 \equiv u^2$.

* The arithmetical consequences of the formulæ in the present paper have been fully developed by the writer, and will be published elsewhere.

† *Œuvres*, (2), p. 240: *Sur les théorèmes de M. Kronecker relatifs aux formes quadratiques.* Hermite's number 3 is misprinted; the left should be $\eta\theta_1 \dfrac{H_1}{\Theta_1}$.

$$(1) \equiv \eta\theta_1 \frac{H}{\Theta} = u \quad ; \quad (2) \equiv \eta\theta_1 \frac{\Theta}{H} = \eta^2\theta_1^2 \frac{1}{u} ;$$

$$(3) \equiv \eta\theta_1 \frac{H_1}{\Theta_1} = \theta_1^2 \frac{v}{w} \quad ; \quad (4) \equiv \eta\theta_1 \frac{\Theta_1}{H_1} = \eta^2 \frac{w}{v} ;$$

$$(5) \equiv \eta\theta \frac{H}{\Theta_1} = \theta^2 \frac{u}{w} \quad ; \quad (6) \equiv \eta\theta \frac{\Theta_1}{H} = \eta^2 \frac{w}{u} ;$$

$$(7) \equiv \eta\theta \frac{H_1}{\Theta} = v \quad ; \quad (8) \equiv \eta\theta \frac{\Theta}{H_1} = \eta^2\theta^2 \frac{1}{v} ;$$

$$(9) \equiv \theta\theta_1 \frac{\Theta_1}{\Theta} = w \quad ; \quad (10) \equiv \theta\theta_1 \frac{\Theta}{\Theta_1} = \theta^2\theta_1^2 \frac{1}{w} ;$$

$$(11) \equiv \vartheta\theta_1 \frac{H_1}{H} = \theta_1^2 \frac{v}{u} \quad ; \quad (12) \equiv \theta\theta_1 \frac{H}{H_1} = \theta^2 \frac{u}{v} ;$$

$$(13) \equiv \theta^2 \frac{\Theta_1 H_1}{\Theta H} = \frac{vw}{u} \quad ; \quad (14) \equiv \theta^2 \frac{\Theta H}{\Theta_1 H_1} = \theta^4 \frac{u}{vw} ;$$

$$(15) \equiv \theta_1^2 \frac{\Theta H_1}{\Theta_1 H} = \theta_1^4 \frac{v}{wu} \quad ; \quad (16) \equiv \theta_1^2 \frac{\Theta_1 H}{\Theta H_1} = \frac{wu}{v} \quad ;$$

$$(17) \equiv \eta^2 \frac{HH_1}{\Theta\Theta_1} = \frac{uv}{w} \quad ; \quad (18) \equiv \eta^2 \frac{\Theta\Theta_1}{HH_1} = \eta^4 \frac{w}{uv} .$$

Cosine series for $(1)^2$, $(2)^2$, $(7)^2$, $(8)^2$, $(9)^2$, $(10)^2$ are given in the *Fundamenta Nova*, sections 41, 42; the remaining squares may be found in several ways, of which the following is sufficiently direct, being based upon the fundamental identities connecting $\mathrm{sn}z$, $\mathrm{cn}z$, $\mathrm{dn}z$. We use this method uniformly in preference to a combination of this and (IV), (V) below, because, although the latter is somewhat shorter, it furnishes fewer useful relations between the squares than does the former. Expressed in terms of u, v, w these fundamental identities, and $k^2 + k'^2 = 1$, become respectively

$$u^2 + v^2 = \eta^4 ; \quad w^2 + u^2 = \theta_1^4 ; \quad w^2 - v^2 = \theta^4 ; \quad \eta^4 + \theta^4 = \theta_1^4 .$$

The first three, for convenience in reductions, are re-written

$$(1)^2 + (7)^2 = \eta^4 ; \quad (9)^2 + (1)^2 = \theta_1^4 ; \quad (9)^2 - (7)^2 = \theta^4 .$$

§ 3. The twelve squares to be calculated fall naturally into three sets

(I) $(3)^2$, $(4)^2$, $(5)^2$, $(6)^2$, $(11)^2$, $(12)^2$;

(II) $(13)^2$, $(16)^2$, $(17)^2$;

(III) $(14)^2$, $(15)^2$, $(18)^2$,

whose members may be expressed linearly in terms of Jacobi's $(1)^2$, $(2)^2$, $(7)^2$, $(8)^2$, $(9)^2$, $(10)^2$ and the constants η^4, θ^4, θ_1^4 as follows

$$\text{(I)}\quad (3)^2-(10)^2=\theta_1^4\left(\frac{v^2-\theta^4}{w^2}\right)=\theta_1^4\left(\frac{v^2-w^2+v^2}{w^2}\right)=2\left(\frac{v\theta_1^2}{w}\right)^2-\theta_1^4,$$

or $$(3)^2-(10)^2=2(3)^2-\theta_1^4,$$

whence $$(3)^2=\theta_1^4-(10)^2;$$

similarly follow $$(4)^2=\eta^4+(8)^2,$$

$$(5)^2=-\theta^4+(10)^2,$$

$$(6)^2=-\eta^4+(2)^2,$$

$$(11)^2=-\theta_1^4+(2)^2,$$

$$(12)^2=-\theta^4+(8)^2.$$

$$\text{(II)}\quad (13)^2-(2)^2=1/u^2(v^2w^2-\eta^4\theta_1^4)=1/u^2\{v^2w^2-(u^2+v^2)(w^2+u^2)\}$$

$$=-w^2-u^2-v^2=-(9)^2-(1)^2-(7)^2,$$

hence $$(13)^2=(2)^2-(1)^2-(7)^2-(9)^2,$$

whence $$(13)^2=-\eta^4+(2)^2-(9)^2;\quad (13)^2=-\theta_1^4+(2)^2-(7)^2.$$

Similarly $$(16)^2=-\theta^4+(1)^2+(8)^2;\quad (16)^2=\eta^4+(8)^2-(9)^2,$$

and $$(17)^2=\theta_1^4-(7)^2-(10)^2;\quad (17)^2=\theta^4+(1)^2-(10)^2.$$

$$\text{(III)}\quad (14)^4=\frac{\theta^8u^2}{v^2w^2}=\frac{(w^2-u^2)^2u^2}{v^2w^2}=\left(\frac{wu}{v}\right)^2-2u^2+\left(\frac{vu}{w}\right)^2,$$

or $(14)^2=(16)^2+(17)^2-2(1)^2$. Substituting for $(16)^2$, $(17)^2$ from (II) and reducing

$$(14)^2=(8)^2-(10)^2.$$

Similarly $$(15)^2=(2)^2-(10)^2,$$

and $$(18)^2=(2)^2+(8)^2.$$

§ 4. *Changes of q into $-q$, of x into $x+\frac{1}{2}\pi$.*

As these transformations are frequently required when using the eighteen quotients, and as they afford convenient checks on the final results of calculations, they may be included here. Let $S(\ )$, $S(\ )^2$ denote the results of replacing x by $x+\frac{1}{2}\pi$ in $(\ )$ and $(\ )^2$ respectively, and $T(\ )$, $T(\ )^2$ the results of replacing q by $-q$ in $(\ )$, $(\ )^2$ respectively. Then, from the elements of elliptic functions,* we have at once

* Cf. *Fundamenta Nova*, section 37.

$$T(\eta^2) = i\eta^2, \quad T(\theta^2) = \theta_1^2, \quad T(\theta_1^2) = \theta^2,$$

$$T(u) = i\theta^2 \frac{u}{w}, \quad T(v) = i\theta_1^2 \frac{v}{w}, \quad T(w) = \theta^2\theta_1^2 \frac{1}{w},$$

$$S(u) = \theta_1^2 \frac{v}{w}, \quad S(v) = -\theta^2 \frac{u}{w}, \quad S(w) = \theta^2\theta_1^2 \frac{1}{w}.$$

The last six may be verified directly from the series given by Hermite,† for (1), (7), (9); (3), (5), (10). From the above we find, on applying $S(\ )$, $T(\ )$ to §2,

(IV) $S(1) = (3)$, $S(2) = (4)$, $S(3) = -(1)$,
$S(4) = -(2)$, $S(5) = (7)$, $S(6) = (8)$,
$S(7) = -(5)$, $S(8) = -(6)$, $S(9) = (10)$,
$S(10) = (9)$, $S(11) = -(12)$, $S(12) = -(11)$,
$S(13) = -(14)$, $S(14) = -(13)$, $S(15) = -(16)$,
$S(16) = -(15)$, $S(17) = -(17)$, $S(18) = -(18)$.

(V) $T(1) = i(5)$, $T(2) = (6)$, $T(3) = i(7)$,
$T(4) = (8)$, $T(5) = i(1)$, $T(6) = (2)$,
$T(7) = i(3)$, $T(8) = (4)$, $T(9) = (10)$,
$T(10) = (9)$, $T(11) = (11)$, $T(12) = (12)$,
$T(13) = (15)$, $T(14) = (16)$, $T(15) = (13)$,
$T(16) = (14)$, $T(17) = -(17)$, $T(18) = (18)$.

Also, if $S\{T(\ )\} \equiv R(\ )$, we have $T\{S(\ )\} = R(\ )$, since x and q are independent, and from (IV), (V)

(VI) $R(1) = i(7)$, $R(2) = (8)$, $R(3) = -i(5)$,
$R(4) = -(6)$, $R(5) = i(3)$, $R(6) = (4)$,
$R(7) = -i(1)$, $R(8) = -(2)$, $R(9) = (9)$,
$R(10) = (10)$, $R(11) = -(12)$, $R(12) = -(11)$,
$R(13) = -(16)$, $R(14) = -(15)$, $R(15) = -(14)$,
$R(16) = -(13)$, $R(17) = (17)$, $R(18) = -(18)$.

These enable us to find $(3)^2$, $(4)^2$, $(5)^2$, $(6)^2$ from Jacobi's series, thus

$$(3)^2 = S(1)^2 = -T(7)^2, \quad (4)^2 = S(2)^2 = T(8)^2,$$
$$(5)^2 = S(7)^2 = -T(1)^2, \quad (6)^2 = S(8)^2 = T(2)^2.$$

† *Loc. cit.*, pp. 242–43.

Also

$$(14)^2 = S(13)^2 = \quad T(16)^2, \quad (15)^2 = S(16)^2 = T(13)^2,$$

which give the first two of (III) from the first two of (II).

§ 5. *Fourier series for* $(1)^2$ *to* $(18)^2$.

In all that follows the summations are with respect to m, n, extending to all odd positive integers m, and to all positive integers n. The constants A, B, and $C \equiv \frac{2K}{\pi} \cdot \frac{2E^1}{\pi}$ are from the *Fundamenta Nova*, section 41.

$$A = 8\Sigma \frac{q^m}{(1-q^m)^2}, \qquad B = 8\Sigma \frac{q^m}{(1+q^m)^2}, \qquad A + B = \eta^4,$$

$$C = \theta^4 + B = \theta_1^4 - A, \quad T(A) = -B, \quad T(B) = -A, \quad T(C) = C,$$

$$\theta^4 = 1 + 8\Sigma \frac{n(-q)^n}{1+q^n}, \quad \theta_1^4 = 1 + 8\Sigma \frac{nq^n}{1+(-q)^n}, \quad \eta^4 = 16\Sigma \frac{mq^m}{1-q^{2m}}.$$

In the following list the series for $(1)^2$, $(2)^2$, $(7)^2$, $(8)^2$, $(9)^2$, $(10)^2$ are written down from Jacobi, and the rest from these by (I), (II), (III). All may be checked at a glance by (IV), (V) or (VI).

$$(1)^2 = A - 8\Sigma \frac{nq^n \cos 2nx}{1-q^{2n}},$$

$$(2)^2 = A + \operatorname{cosec}^2 x - 8\Sigma \frac{nq^{2n} \cos 2nx}{1-q^{2n}},$$

$$(3)^2 = A - 8\Sigma \frac{n(-q)^n \cos 2nx}{1-q^{2n}},$$

$$(4)^2 = A + \sec^2 x - 8\Sigma \frac{n(-1)^n q^{2n} \cos 2nx}{1-q^{2n}},$$

$$(5)^2 = B + 8\Sigma \frac{n(-q)^n \cos 2nx}{1-q^{2n}},$$

$$(6)^2 = -B + \operatorname{cosec}^2 x - 8\Sigma \frac{nq^{2n} \cos 2nx}{1-q^{2n}},$$

$$(7)^2 = B + 8\Sigma \frac{nq^n \cos 2nx}{1-q^{2n}},$$

$$(8)^2 = -B + \sec^2 x - 8\Sigma \frac{n(-1)^n q^{2n} \cos 2nx}{1-q^{2n}},$$

$$(9)^2 = C + 8\Sigma \frac{nq^n \cos 2nx}{1-q^{2n}},$$

$$(10)^2 = C + 8\Sigma \frac{n(-q)^n \cos 2nx}{1-q^{2n}},$$

$$(11)^2 = -C + \operatorname{cosec}^2 x - 8\Sigma \frac{nq^{2n} \cos 2nx}{1-q^{2n}},$$

$$(12)^2 = -C + \sec^2 x - 8\Sigma \frac{n(-1)^n q^{2n} \cos 2nx}{1-q^{2n}},$$

$$(13)^2 = -B - C + \operatorname{cosec}^2 x - 8\Sigma \frac{nq^n \cos 2nx}{1-q^n},$$

$$(14)^2 = -B - C + \sec^2 x - 8\Sigma \frac{n(-q)^n \cos 2nx}{1-q^n},$$

$$(15)^2 = A - C + \operatorname{cosec}^2 x - 8\Sigma \frac{n(-q)^n \cos 2nx}{1-(-q)^n},$$

$$(16)^2 = A - C + \sec^2 x - 8\Sigma \frac{nq^n \cos 2nx}{1-(-q)^n},$$

$$(17)^2 = A - B - 32\Sigma \frac{nq^{2n} \cos 4nx}{1-q^{4n}},$$

$$(18)^2 = A - B + 4\operatorname{cosec}^2 2x - 32\Sigma \frac{nq^{4n} \cos 4nx}{1-q^{4n}}.$$

All eighteen may be derived by means of S, T, R from the set of seven—$(1)^2$, $(2)^2$, $(9)^2$, $(11)^2$, $(13)^2$, $(17)^2$, $(18)^2$. We next find similar developments for certain products of (1) to (18), § 2. They are of considerable importance in arithmetical applications, as they afford a kind of connecting link between Hermite's second and third sets of twelve non-doubly periodic functions each.

§ 6. *x-derivatives of the eighteen quotients.*

Denoting by $(\ )'$ the x-derivative of $(\ )$, we have, from § 1, since $z' = \theta_1^2$,

(VII) $$u' = vw, \quad v' = -wu, \quad w' = -uv,$$

as the equivalents respectively of

$$\frac{d}{dz}\operatorname{sn} z = \operatorname{cn} z \operatorname{dn} z, \quad \frac{d}{dz}\operatorname{cn} z = -\operatorname{sn} z \operatorname{dn} z, \quad \frac{d}{dz}\operatorname{dn} z = -k^2 \operatorname{sn} z \operatorname{cn} z.$$

Applying (VII) on (1) to (18), § 2, we find easily, after all reductions by means of the fundamental identities,

(VIII)

$$(1)' = (7)\,(9), \quad (7)' = -(1)\,(9), \quad (9)' = -(1)\,(7),$$
$$(2)' = -(6)\,(11), \quad (6)' = -(2)\,(11), \quad (11)' = -(2)\,(6),$$
$$(3)' = -(5)\,(10), \quad (5)' = (3)\,(10), \quad (10)' = (3)\,(5),$$
$$(4)' = (8)\,(12), \quad (8)' = (4)\,(12), \quad (12)' = (4)\,(8),$$
$$(13)' = -(6)^2 - (7)^2, \quad (14)' = (5)^2 + (8)^2, \quad (15)' = -(2)^2 + (3)^2,$$
$$(16)' = (9)^2 + (12)^2, \quad (17)' = -(1)^2 + (3)^2, \quad (18)' = -(2)^2 + (4)^2.$$

Directly from these and § 2 there are the alternatives

$$(2)' = -(2)\,(13), \quad (8)' = (8)\,(16), \quad (10)' = (10)\,(17).$$

It is easy to check the above by differentiation of the ten equations (IX) written down from § 2. These are themselves useful in the applications.

(IX)
$$(1)\,(2) = (3)\,(4) = \eta^2\theta_1^2, \quad (13)\,(14) = \theta^4,$$
$$(5)\,(6) = (7)\,(8) = \eta^2\theta^2, \quad (15)\,(16) = \theta_1^4,$$
$$(9)\,(10) = (11)\,(12) = \theta^2\theta_1^2, \quad (17)\,(18) = \eta^4,$$
$$(13)\,(14) + (17)\,(18) = (15)\,(16),$$

the last being equivalent to $\theta^4 + \eta^4 = \theta_1^4$.

§ 7. *Series for the products in* § 6.

Taking the x-derivatives of Hermite's series for (1) to (12), as given on pp. 242–3 of the work cited, from (VIII) we have

$$(7)\,(9) = 4\Sigma \frac{m\sqrt{(q^m)}\cos mx}{1-q^m},$$

$$(6)\,(11) = \operatorname{cosec} x \cot x - 4\Sigma \frac{mq^m \cos mx}{1-q^m} = (2)\,(13),$$

$$(5)\,(10) = 4\Sigma \frac{(-1)^{\frac{1}{2}(m-1)} m\sqrt{(q^m)}\sin mx}{1-q^m},$$

$$(8)\,(12) = \sec x \tan x - 4\Sigma \frac{(-1)^{\frac{1}{2}(m-1)} mq^m \sin mx}{1-q^m},$$

$$(3)\,(10) = 4\Sigma \frac{(-1)^{\frac{1}{2}(m-1)} m\sqrt{(q^m)}\cos mx}{1+q^m},$$

$$(2)\,(11) = \operatorname{cosec} x \cot x + 4\Sigma \frac{mq^m \cos mx}{1+q^m},$$

$$(1)\,(9) = 4\Sigma \frac{m\sqrt{(q^m)}\sin mx}{1+q^m},$$

$$(4)\,(12) = \sec x \tan x + 4\Sigma \frac{(-1)^{\frac{1}{2}(m-1)} m q^m \sin mx}{1+q^m} = (8)\,(16).$$

$$(1)\quad (7) = 8\Sigma \frac{nq^n \sin 2nx}{1+q^{2n}},$$

$$(3)\quad (5) = -8\Sigma \frac{(-1)^n nq^n \sin 2nx}{1+q^{2n}} = (10)\,(17),$$

$$(2)\quad (6) = \operatorname{cosec}^2 x + 8\Sigma \frac{nq^{2n} \cos 2nx}{1+q^{2n}},$$

$$(4)\quad (8) = \sec^2 x + 8\Sigma \frac{(-1)^n nq^{2n} \cos 2nx}{1+q^{2n}}.$$

We notice that these fall into three sets of four each according to the forms of the denominators $1-q^m$, $1+q^m$, $1+q^{2m}$; also, that in each set there are two even and two odd functions. All twelve at the left may be generated by S, T, R from the set of four: (7) (9), (6) (11), (1) (7), (2) (6). The set of twelve in § 7, being multiplied together in pairs, gives six identities between series which are extremely fertile in arithmetical consequences. The third pair in each is from (IX):

$$\begin{aligned}
-(1)'\,(2)' = (6)\,(7)\,(9)\,(11) &= \eta^2 \theta_1^{\,2}(13)^2 \\
&= (1)\,(2)\,(13)^2 = (3)\,(4)\,(13)^2, \\
-(3)'\,(4)' = (5)\,(8)\,(10)\,(12) &= \eta^2 \theta_1^{\,2}(14)^2 \\
&= (3)\,(4)\,(14)^2 = (1)\,(2)\,(14)^2, \\
-(5)'\,(6)' = (2)\,(3)\,(10)\,(11) &= \eta^2 \theta^4 (15)^2 \\
&= (5)\,(6)\,(15)^2 = (7)\,(8)\,(15)^2, \\
-(7)'\,(8)' = (1)\,(4)\,(9)\,(12) &= \eta^2 \theta^2 (16)^2 \\
&= (7)\,(8)\,(16)^2 = (5)\,(6)\,(16)^2, \\
-(9)'\,(10)' = (1)\,(3)\,(5)\,(7) &= \theta^2 \theta_1^{\,2}(17)^2 \\
&= (9)\,(10)\,(17)^2 = (11)\,(12)\,(17)^2, \\
-(11)'\,(12)' = (2)\,(4)\,(6)\,(8) &= \theta^2 \theta_1^{\,2}(18)^2 \\
&= (11)\,(12)\,(18)^2 = (9)\,(10)\,(18)^2.
\end{aligned}$$

§ 8. We may notice here, without pursuing the subject further, that the Fourier expansions for $(1)^n$ to $(12)^n$, for $n>2$, and in particular for $n=3$, 4, may be found by applying S, T, R to the series in section 43 of the *Fundamenta Nova*. The series for $(13)^n$ to $(18)^n$ would seem to require more

complicated considerations. The cases $n=3$, 4, give important arithmetical theorems; when $n>4$, the results of arithmetically paraphrasing the elliptic function formulæ no doubt are important, but they are so complex as to be almost unintelligible. Similar remarks apply to products of more than four factors (1) to (18) of § 2.

We will conclude with a set of triple products.

§ 9. *Fourier series for four triple products.*

Denoting the x-derivative of $(1)^2$ by $(1^2)'$, and similarly for the derivates of $(2)^2$ to $(12)^2$, we have, from VIII,

$$(1^2)' = -(7^2)' = -\ (9^2)' = \ \ 2\,(1)\,(7)\ \ (9),$$
$$(2^2)' = \ \ (6^2)' = \ \ (11^2)' = -\,2\,(2)\,(6)\,(11),$$
$$(3^2)' = -(5^2)' = -(10^2)' = -\,2\,(3)\,(5)\,(10),$$
$$(4^2)' = \ \ (8^2)' = \ \ (12^2)' = \ \ 2\,(4)\,(8)\,(12),$$

whence, taking the x-derivatives of $(1)^2$, $(2)^2$, $(3)^2$, $(4)^2$ of § 5,

$$(1)\,(7)\ \ (9) = 8\Sigma\,\frac{n^2q^n\sin 2nx}{1-q^{2n}},$$

$$(2)\,(6)\,(11) = \operatorname{cosec}^2x\cot x - 8\Sigma\,\frac{n^2q^{2n}\sin 2nx}{1-q^{4n}},$$

$$(3)\,(5)\,(10) = -\,8\Sigma\,\frac{n^2(-q)^n\sin 2nx}{1-q^{2n}},$$

$$(4)\,(8)\,(12) = \sec^2x\tan x + 8\Sigma\,\frac{n^2(-1)^nq^{2n}\sin 2nx}{1-q^{2n}}.$$

THE q-INTEGRAL ANALOGOUS TO BOREL'S INTEGRAL.

By *Rev. F. H. Jackson, Sc.D.*

Introduction.

In a paper on "q-Definite Integrals" (*Quarterly Journal of Mathematics*, vol. xli., 163, 1910) the author introduced an operation termed q-integration, with the notation

$$\mathbf{S}\,\Phi(x)\,d(qx),$$

which in case $q=1$ and $\Phi(x)$ possessed an integral reduced to

$$\int\Phi(x)\,dx.$$

For example

$$\int_0^\infty \frac{x^{a-1}}{1-x}\, d(qx)$$

was shown to be $\Gamma_q(\tfrac{1}{2})^2 / \sigma_q(a)$,

in which σ denoted a doubly periodic sigma-function and Γ_q the basic gamma-function. In case $q=1$ the integral reduced to

$$\int_0^\infty \frac{x^{a-1}}{1-x}\, dx = \frac{\pi}{\sin \pi a}.$$

Other examples of the q-integrals, chiefly in connection with the basic gamma-function, were given.

In this paper, which is a continuation of the former paper, certain q-integrals will be determined, and the connection between the q-definite-integral

$$\int_0^\infty E(-x)\, u(x)\, d(qx) \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and a generalization of Euler's transformation for power series investigated. This integral, when $q=1$, reduces to Borel's

$$\int_0^\infty e^{-x} u(x)\, dx,$$

$u(x)$ in (1) denotes the associated function

$$\Sigma u_n \frac{x^n}{[n]!},$$

in which $[n]$ denotes the basic number

$$(q^n-1)/(q-1).$$

In this paper the infinite q-integral

$$\int_0^\infty E(-x)\, x^n\, d(qx) = \Gamma_q(n+1)$$

will be obtained directly from $\Gamma_q(n+1)$ defined as the limit

$$\lim_{\kappa=\infty} \frac{[\kappa]^n}{[n+1][n+2]\ldots[n+\kappa]}\,[\kappa]^n.$$

This is necessary to complete the work on p. 199, *Quarterly Journal of Mathematics*, vol. xli.

Other interesting q-integrals are

Ex. i. $$\int x^n \log x = \frac{x^{n+1}}{[n+1]} \log x - \frac{(qx)^{n+1}}{[n+1]} \frac{\log q}{q-1} + \text{const.}$$

Ex. ii. $$\int_0^1 \frac{x^n \log x}{1-x}\, d(qx) = \frac{\log q}{q-1} \left\{ \frac{q^{n+1}}{[n+1]^2} + \frac{q^{n+2}}{[n+2]^2} + \frac{q^{n+3}}{[n+3]^2} + \ldots \right\}.$$

I.

As in previous papers

$$[n] \text{ denotes } (q^n-1)/(q-1);$$

$$(1+x)^{(n)} = \operatorname*{limit}_{\kappa=\infty} \frac{(1+x)\,(1+qx)\ldots(1+q^{\kappa}x)}{(1+q^n x)\,(1+q^{n+1}x)\ldots(1+q^{n+\kappa}x)}; \qquad |q| \begin{matrix}\leq 1\\ \geq 1\\ \neq\end{matrix}$$

$$= 1 + [n]\,x + q\,\frac{[n]\,[n-1]}{[2]\,!}\,x^2 + \ldots;$$

which is termed the q-binomial series, the general term being

$$q^{n(n-1)/2}\,\frac{[n]\,!}{[r]\,!\,[n-r]\,!}\,x^r.$$

The q-exponential series is defined as

$$E_q(x) = \operatorname*{limit}_{n\to\infty} \left\{1 + x + \frac{x^2}{[2]\,!} + \frac{x^3}{[3]\,!} + \ldots + \frac{x^n}{[n]\,!}\right\},$$

in connection with which we also have

$$E_{1/q}(x) = \operatorname*{limit}_{n\to\infty} \left\{1 + x + q\,\frac{x^2}{[2]\,!} + \ldots + q^{n(n-1)/2}\frac{x^n}{[n]\,!}\right\}.$$

The second of these series may be derived from the first by inversion of the base q. It is easily seen also that they may be obtained as limiting cases of the q-binomial series in analogy with

$$\exp(x) = \left\{\operatorname*{limit}_{n\to\infty}\left(1 + \frac{1}{n}\right)^n\right\}^x.$$

We may remark, in passing, that it is not difficult to obtain from them a q-extension of Bernouilli's functions $\mathbf{S}(x)$; such that

$$\mathbf{S}_q(x, 2) - \mathbf{S}_q(x, 1) = [n]\,(1+x)^{(n-1)},$$

but this is outside the scope of this paper. Conditions for the convergence of the series are easily established, and as the series E_q and $E_{1/q}$ have to be used in conjunction, it is well to note that they have a region of convergence in common. The following elementary properties of these functions are easy to establish, and, as I have done this in former papers, it will be enough to state them here and number them for easy reference in subsequent work.

I. $(1-x)^{(m)} \times (1-q^m x)^{(n)} = (1-x)^{(m+n)}$;

II. $E_q(x)\,E_{1/q}(y) = 1 + (x+y) + \dfrac{(x+y)\,(x+qy)}{[z]\,!} + \ldots + \dfrac{(x+y)^{(r)}}{[r]\,!} + \ldots;$

III. $E_q(x) \times E_{1/q}(-x) = 1$;

IV. $E_q(x)\ E_q(-x) = E_{q^2}\left(x^2 \frac{q-1}{q+1}\right)$;

V. $x^n E_q(-ax)$; and $x^n E_{1/q}(-ax)$;

both vanish for $x=0$, and $x=\infty$, provided $a \gtrless 1$ and n is positive.

II.

Δ denotes an operative symbol which, when applied to $\phi(x)$, gives us

$$\Delta\phi(x) = \frac{\phi(qx) - \phi(x)}{qx - x},$$

the quotient of two differences. Let us denote $qx - x$ by $d(qx)$, and in analogy with the notation of ordinary integrations we shall use $\mathbf{S}$ as the sign of q-integration, thus

$$\mathbf{S}\phi(x)\,d(qx).$$

For the expression of Δ in terms of $x\frac{d}{dx}$ and the generalization of Boole's operational equation, see *Messenger of Mathematics*, vol. xxxviii., p. 57, where also

$$\lim_{q=1} \mathbf{S}\phi(x)\,d(qx)$$

is shown as equal to $\int\phi(x)\,dx$.

Table of Differences and q-Integrals.

With Δ and $\mathbf{S}$ operations defined as in the preceding section, we easily form the following differences and q-integrals, all of which will be needed in subsequent work. We note that in the case of the integrals the constant of integration is not an absolute constant, as in ordinary integration, but is what was termed by Boole a "periodic constant." In the case of q-integrals the "constant" will be in general any solution of $f(x) = f(qx)$. Such would be

$$\exp\left\{a\frac{2\pi i \log x}{\log q}\right\}.$$

Throughout this paper the constant will be always taken in its simplest form, namely, absolute, as in ordinary integration.

(α) $\Delta ax^n = a\,[n]\,x^{n-1}$,

$$\mathbf{S}\,ax^n d(qx) = \frac{a}{[n+1]}x^{n+1} + \text{const},$$

$$(\beta)\quad \Delta(1+ax)^{(n)} = a[n](1+aqx)^{(n-1)},$$

$$\mathbf{S}(1+aqx)^{(n)}\,d(qx) = \frac{1}{a[n+1]}(1+ax)^{(n+1)},$$

$$\Delta E_q(ax) = a E_q(ax),$$

$$(\gamma)\quad \Delta E_q(ax) = a E_q(ax),$$

$$(\delta)\quad \Delta E_{1/q}(ax) = a E_{1/q}(aqx),$$

$$\mathbf{S} E_{1/q}(aqx)\,d(qx) = a^{-1} E_{1/q}(ax),$$

$$\mathbf{S} E_{1/q}(aq^n x)\,d(qx) = a^{-1} E_{1/q}(aq^{n-1}x),$$

all of which are analogous to

$$\frac{d}{dx} e^{ax} = a e^{ax},$$

$$\int e^{ax}\,dx = a^{-1} e^{ax} + \text{const.}$$

(ϵ) Since $\Delta u_x v_x = (u_{qx} v_{qx} - u_x v_x)/(qx - x)$,

$$= u_x' v_x + u_{qx} v_x',$$

where $u_x' = \Delta u_x$, etc.,

we have a formula for integration by parts

$$\mathbf{S} v_x u_x'\,d(qx) = u_x v_x + \mathbf{S} u_{qx} v_x'\,d(qx),$$

whence

$$\mathbf{S} E_q(bx) E_{1/q}(aqx) = \frac{1}{(a+b)} E_q(bx) E_{1/q}(ax).$$

This integral would also be obtained by forming the series

$$E_q(bx) E_{1/q}(ax) = 1 + (a+b)x + \frac{(a+b)(a+bq)}{[2]!}x^2 + \ldots,$$

and taking the q-differences term by term.

(ζ) By part integration we have

$$\mathbf{S} x^n E_q(-bx) E_{1/q}(-aqx)\,d(qx)$$

$$= -\frac{x^n}{a+b} E_q(-bx) E_{1/q}(-ax) + \mathbf{S}\frac{[n]}{a+b} x^{n-1} E_q(-bqx) E_{1/q}(-aqx).$$

(η) If we take the integral in the preceding (ζ) between limits 0 and ∞, we have

$$\overset{\infty}{\mathbf{S}}_0 x^n E_q(-bx) E_{1/q}(-aqx)\,d(qx)$$

$$= \frac{[n]}{b+a} \overset{\infty}{\mathbf{S}}_0 x^{n-1} E_q(-bqx) E_{1/q}(-aqx)\,d(qx)$$

(n positive).

The analogous form when $q = 1$ is of course

$$\int_0^\infty x^n e^{-(a+b)x} dx = \frac{n}{a+b} \int_0^\infty x^{n-1} e^{-(a+b)x} dx.$$

When n is an integer the integral becomes, by repetition, finally equal to

$$\frac{[n]!}{(a+b)(a+bq)\ldots(a+bq^{n-1})} \quad \ldots\ldots\ldots\ldots\ldots\ldots (A).$$

In general, when n is not an integer, we should have the function

$$\frac{\Gamma([n+1])}{a^n\{1+(b/a)\}^{(n)}}.$$

III.

At this point it will be well to obtain the infinite q-integral expression for the basic gamma-function directly from the definition of the function

$$\Gamma_q(y+1) = \lim_{n\to\infty} \frac{[n]!}{[y+1][y+2]\ldots[y+n]}\,[n]^y\,q \gtrless 1.$$

Consider $\quad \mathbf{S}_0^1 x^{y+s}(1-qx)^{(n-s)}\,d(qx) = I_s,$

using the formula for integration by parts in which

$$u_x' = (1-qx)^{(n-s)},$$

therefore $\quad u_x = -(1-x)^{(n-s+1)}/[n-s+1],$

$$v_x = x^{y+s},$$

$$v_x' = [y+s]x^{y+s-1},$$

we have

$$\mathbf{S}x^{y+s}(1-qx)^{(n-s)}\,d(qx) = -x^{y+s}(1-x)^{(n-s+1)}/[n-s+1] + \mathbf{S}\frac{[y+s]}{[n-s+1]}x^{y+s-1}(1-qx)^{(n-s+1)},$$

taking the integrals between limits 0 and 1 we have

$$I_s = \frac{[y+s]}{[n-s+1]}\,I_{s-1},$$

also $\quad I_n = \mathbf{S}_0^1 x^{y+n}\,d(qx) = 1/[y+n+1],$

whence $\quad \frac{I_0}{I_n} = \frac{[n]!}{[y+1][y+2]\ldots[y+n]},$

whence $\quad \Gamma_q(1+y) = \lim [n]^y[n+y+1]I_0$

$$= \lim_{n\to\infty}[n]^y[n+y+1]\,\mathbf{S}_0^1 x^y(1-qx)^{(n)}\,d(qx).$$

Change the variable x by substituting $x = \frac{t}{[n]}$ $(q>1)$. Then

$$d(qx) = \frac{1}{n}\,d(qt)\left(1-q\frac{t}{[n]}\right)^{(n)} = 1 - qt + q^3\frac{[n][n-1]}{[n]^2[2]}t^2 - \ldots.$$

In the limit $n \to \infty$ the series on the right becomes $(q > 1)$

$$1 - qt + \frac{q^2 t^2}{[2]!} - \frac{q^3 t^3}{[3]!} + \ldots = E_q(-qt),$$

so that

$$\Gamma_q(1+y) = \operatorname*{limit}_{n\to\infty} \frac{[n+y+1]}{[n]} \overset{(n)}{\underset{0}{\mathbf{S}}} t^y E_q(-qt)\, d(qt)$$

$$= q^{y+1} \overset{\infty}{\underset{0}{\mathbf{S}}} E_q(-qt) \cdot t^y d(qt).$$

If for t we put x/q, this simplifies to

$$\overset{\infty}{\underset{0}{\mathbf{S}}} E_q(-x) x^y d(qx).$$

IV.

The q-integral analogous to Borel's integral and its connection with the transformation of q-power series. Consider a power series Σu_n, where $u_n = a_n t^n$, and a_n is real and positive, while t is real and < 1. Then introducing coefficients

$$(A) \quad \begin{cases} b_0 = a_0, \\ b_1 = a_0 - a_1 = (1-D)\, a_0, \\ b_2 = a_0 - [2]\, a_1 + q a_2 = (1-D)(1-qD)\, a_0, \end{cases}$$

and, in general,

$$b_r = a_0 - [r]\, a_1 + q \frac{[r][r-1]}{[2]!} a_2 - \ldots (-)^r q^{r(r-1)/2} a_n = (1-D)^{(r)} a_0.$$

Then, in analogy with Borel's notation, take

$$u(x) = \Sigma u_n \frac{x^n}{[n]!}.$$

From (A) above, by easy induction, we obtain

$$\begin{aligned} a_0 &= b_0, \\ a_1 &= b_0 - b_1, \\ a_2 &= q b_0 - [2]\, b_1 + b_2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \\ a_n &= q^{n(n-1)/2} b_0 - q^{(n-1)(n-2)/2} [n] b_0 + \ldots (-1)^n b_n. \end{aligned}$$

so that $u(x)$ may be written

$$\begin{aligned} u(x) = b_0 & \\ &+ (b_0 - b_1) xt, \\ &+ (q b_0 - [2]\, b_1 + b_2) \frac{x^2 t^2}{[2]!}, \\ &+ (q^3 b_0 - q\,[3]\, b_1 + [3]\, b_2 - b_3) \frac{x^3 t^3}{[3]!}, \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \end{aligned}$$

Assuming the absolute convergence of $u(x)$, we can sum this series by columns, and so obtain

$$b_0 E_{1/q}(xt) - b_1 xt E_{1/q}(xt) + b_2 \frac{x^2t^2}{[2]!} E_{1/q}(xt) - \dots$$

Hence the q-integral analogous to Borel's $\int_0^\infty e^{-x} u(x)\,dx$, namely,

$$\mathop{\mathrm{S}}_{0}^{\infty} E_q(-x)\,u(x)\,d(qx) \quad \dots\dots\dots\dots\dots\dots (B)$$

becomes

$$\mathop{\mathrm{S}}_{0}^{\infty} E_q(-x)\,E_{1/q}(xt)\left\{b_0 - b_1 xt + b_2 \frac{x^2t^2}{[2]!} - \dots\right\} d(qx).$$

Integrating term by term and noting that

$$\mathop{\mathrm{S}}_{0}^{\infty} x^n E_q(-x)\,E_{1/q}(tx)\,d(qx),$$

when n is positive and integral, is

$$\frac{[n]!}{\{1-(t/q)\}\{q-(t/q)\}\dots\{q^{n-1}-(t/q)\}},$$

we have finally the series

$$\frac{b_0}{1-(t/q)} - \frac{b_1}{\{1-(t/q)\}\{q-(t/q)\}} + \dots,$$

or expressed in terms of $a_0, a_1, a_2, \dots$,

$$\frac{a_0}{1-(t/q)} - \frac{\Delta a_0 t}{\{1-(t/q)\}\{q-(t/q)\}} + \frac{\Delta^{(2)} a_0 t^2}{\{1-(t/q)\}\{q-(t/q)\}\{q^2-(t/q)\}} - \dots$$
$$\dots\dots\dots (C),$$

where

$$\Delta a_0 = a_0 - a_1 = (1-D)\,a_0,$$
$$\Delta^2 a_0 = (1-D)(1-qD)\,a_0,$$
$$\Delta^n a_0 = (1-D)^{(n)} a_0,$$
$$\dots\dots\dots\dots\dots\dots .$$

This transformation of the power series $\Sigma a_n t^n$ into (C) reduces to Euler's well-known transformation, in case we put $q=1$. The connection between Borel's integral and Euler's series is discussed by Dr. Bromwich*. It will be seen that the corresponding connection for q-series worked out in this paper is on lines parallel to Dr. Bromwich's work. In a paper† I have shewn how the q-integral (B), which corresponds in basic-function theory to Borel's integral in ordinary function theory, may be used to sum q-series.

* *Theory of Infinite Series*, Ed. 1908, pp. 303–5.
† *Proceedings of Royal Society of Edinburgh*, vol. xxx., part v., no. 22 (1910).

ON THE CLASS NUMBER FOR DEFINITE TERNARY QUADRATICS.

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After Dirichlet had found the number of classes of binary quadratics with a given determinant, Eisenstein,* as he tells us, devoted himself unsuccessfully for seven years to the similar problem for the ternary quadratics. His efforts, however—as is often the case in investigations dealing with difficult problems in the theory of numbers—were, as he says, not altogether in vain. For he developed a fairly complete account of the arithmetical theory† of the ternary quadratic. He shewed that, unlike the binary quadratic which has only one invariant for linear substitutions of determinant unity, the ternary quadratic has two arithmetical invariants. Thus for the form $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$, the common factor Ω of $bc-f^2$, $ca-g^2$, ..., $af-gh$, ... is an arithmetical invariant. Another one is an integer Δ given by

$$abc+2fgh-af^2-bg^2-ch^2=\Omega^2\Delta,$$

and the complex of forms with given invariants Ω, Δ is referred to as the order (Ω, Δ). This classification suffices if Ω and Δ are odd and the ternary quadratics are definite forms (the principal case considered by Eisenstein), and was extended by H. J. S. Smith to the general quadratic form with any number of variables. Eisenstein also defined the generic characters of the forms and subdivided the order into its genera.

In the case of a definite form he introduced the very important conception of the weight of a form, which he defined as the reciprocal of the number of linear substitutions with integral coefficients and determinant positive unity, which change the form into itself. He defined the weight of a complex of forms as the sum of the weights of the non-equivalent forms of the complex, and when Ω and Δ are both odd, he gave, without proof, formulæ for the weight of a given *genus*, and also of a given order.‡ These were probably obtained

* Uber die Vergleichung von solchen ternären quadratischen Formen, welche verschiedene Determinanten haben. Monatsbericht der Königl. Preuss. *Akademie der Wissenschaften zu Berlin* (1852), p. 369, hereafter referred to as Monatsberichte (1852).

† Neue Theoreme der höheren Arithmetik, *Crelle*, vol. xxxv., p. 117.

‡ In his paper, "Uber reducirte positive ternäre quadratische Formen," *Crelle* (1851), vol. xli., p. 152, he gives some results when Ω and Δ are not both odd. This paper will be referred to as *Crelle* (1851).

by purely arithmetical considerations. They were proved by H. J. S. Smith* by applying the principles introduced by Dirichlet and extended to include all cases.

Finally, Eisenstein, after three years more of concentrated effort upon the subject, discovered a simple method of dealing with the class number for all quadratic forms.† The method for definite forms was to consider the number of classes of forms for which δ the number of positive automorphics (automorphics of which the determinant is a positive unit) has a given value. If we call $\sigma_1, \sigma_2, \sigma_4, \ldots$, the number of classes for which $\delta = 1, 2, 4, 6, 8, 12, 24$, and no other values of δ occur, the weight M of a given complex of forms is by definition

$$M = \sigma_1 + \tfrac{1}{2}\sigma_2 + \tfrac{1}{4}\sigma_4 + \tfrac{1}{6}\sigma_6 + \tfrac{1}{8}\sigma_8 + \tfrac{1}{12}\sigma_{12} + \tfrac{1}{24}\sigma_{24}.$$

The class number H of the complex is

$$H = \sigma_1 + \sigma_2 + \sigma_4 + \sigma_6 + \sigma_8 + \sigma_{12} + \sigma_{24},$$

so that

$$H = M + \tfrac{1}{2}\sigma_2 + \tfrac{3}{4}\sigma_4 + \tfrac{5}{6}\sigma_6 + \tfrac{7}{8}\sigma_8 + \tfrac{11}{12}\sigma_{12} + \tfrac{23}{24}\sigma_{24},$$

where the value of M is known for a given *genus* or order. The values of $\sigma_4, \sigma_6, \ldots$, depend upon the factors of the determinant D of the forms, while σ_2 depends upon the class number for definite binary quadratics of determinants which are divisors of $2D$; and if the ternary forms are restricted to a given *genus*, this defines the generic character of the binary forms.

These results, he says, depend upon the theorem that every definite ternary quadratic which possesses more than one positive automorphic is equivalent to a form $f(x, y, z)$, where either

$$f(x, y, z) = ax^2 + by^2 + 2fyz + cz^2,$$

or

$$2f(x, y, z) = a(2x + y)^2 + by^2 + 2fyz + cz^2,$$

and a is a divisor of D, $2D$ respectively, while $by^2 + 2fyz + cz^2$ is a binary quadratic of determinant $-D/a$, $-2D/a$ respectively. Conversely, these ternary forms clearly possess more than one automorphic.

As an illustration he then takes the case when D is a prime number greater than 3, and considers the number of

* "On the orders and genera of ternary quadratic forms," *Collected Works*, vol. i., p. 454 *et seq.* They are also given by Bachman in his "Die Arithmetik der Quadratischen Formen," forming Part 4 of his *Zahlentheorie*, p. 196. This book will be referred to hereafter as Bachman.

† *Monatsberichte*, 369. The only reference to indefinite forms that I can find is in *Crelle* (1851), p. 168, where he says, "The class number for indefinite forms is much simpler than for definite forms, at least when the determinant is an odd number without squared factors. The number of classes in this case seems to be 2^μ if μ is the number of odd prime factors dividing the determinant."

classes in the *genus** defined by $\left(\frac{-F}{D}\right)=1$. He states the values of σ_2, σ_4, ..., and so finds the number of classes in the given *genus*. In *Crelle's Journal* (1851), p. 155, he states that for all values of D the number of classes of definite ternary quadratics of determinant D depends upon the number of classes of definite binary quadratics of determinants which are divisors of D or of $2D$. Calling $H(D)$ the number of properly primitive non-equivalent ternary forms of determinant D, $H'(D)$ the corresponding number for improperly primitive forms, and $h(d)$, $h'(d)$ respectively the number of properly primitive and improperly primitive non-equivalent binary forms of determinant $-d$, then in the simplest cases when the determinant has no squared factors,

$$H(P) = \tfrac{1}{4}\{\Sigma h(d) + \Sigma h'(d) + \Sigma h(2d)\} + \tfrac{1}{12}(P+\lambda),$$
$$H(2P) = \tfrac{1}{2}\Sigma h(d) + \tfrac{1}{4}\Sigma h(2d) + \tfrac{1}{8}(P+\nu),$$
$$H'(2P) = \tfrac{1}{4}\{\Sigma h(d) + \Sigma h'(d)\} + \tfrac{1}{24}(P+\rho),$$

where P is the product of different odd prime numbers, d refers to all divisors of P, excluding unity, and the summation refers to the different values of d,

$$\lambda = 9 + 2\left(\frac{P}{3}\right), \quad \nu = 6 + \left(\frac{-1}{P}\right), \quad \rho = 6 - 3\left(\frac{-1}{P}\right) - 4\left(\frac{P}{3}\right),$$

where $\left(\frac{P}{3}\right)$ and $\left(\frac{-1}{P}\right)$ are symbols of quadratic reciprocity. For all values of D similar formulæ exist, but he had not as yet reduced them to a form sufficiently simple for publication. Unfortunately Eisenstein died shortly afterwards (at the early age of 29), and his important researches on these subjects were never published. Bachman† promises us some account of this subject in the second part of his "Die Arithmetik der Quadratischen Formen," but this part has not as yet appeared.

By following the method indicated by Eisenstein, I have been able to prove his results when D has no squared factors. The same method applies for all values of D, but even in the simplest case the investigation requires attention to a great number of details with respect to which it is very easy to fall into error. It is noteworthy that, assuming the formula for the weight of an order, the work involves only elementary processes. As it is very likely that Eisenstein found the weight arithmetically (he has given a method in a particular

* F stands for the contravariant of the ternary forms.
† Bachman, p. 271.

case*), it seems that the derivation of the formulæ for the class number of ternary quadratics is a much simpler process than the corresponding process for binary quadratics.

Considering now only definite forms, it is evident that the automorphics† of a ternary quadratic form a group of finite order. It will be shewn that this group either reduces to the identical operation or is of even order. Hence if a ternary form possesses more than one automorphic, there exists an automorphic which is of the second order, that is to say, it is identical with its inverse. Upon this fact depends the success of the investigation.

It is well known that if the substitution S

$$x' = l_1 x + m_1 y + n_1 z$$
$$y' = l_2 x + m_2 y + n_2 z$$
$$z' = l_3 x + m_3 y + n_3 z$$

is an automorphic of a ternary quadratic,‡ then the equation

$$\begin{vmatrix} l_1 - k, & m_1, & n_1 \\ l_2, & m_2 - k, & n_2 \\ l_3, & m_3, & n_3 - k \end{vmatrix} = 0$$

is a reciprocal equation of which one root is given by $k = 1$. This is tantamount to saying that we can find integers p, q, r without a common factor, so that by putting $\xi = px + qy + rz$, then for the substitution S, $\xi' = \xi$. Hence the ternary form is equivalent to another which has an automorphic T given by

$$x' = x$$
$$y' = lx + my + nz$$
$$z' = px + qy + rz.$$

The substitution U given by $y' = my + nz$, $z' = qy + rz$ is a positive automorphic of a definite binary quadratic, so that U is of order 1, 2, 3, 4 or 6. If U is of order 1, T is the identical substitution, and if U is of order 2, 4, 6, T is of an even order. If U is of order 3, the binary quadratic is derived from the form $2y^2 + 2yz + 2z^2$, and it can be shewn that the ternary form admits of an automorphic of order 2. For, taking U to be the substitution $y' = -y - z$, $z' = y$, T is the substitution

* *Monatsberichte*, p. 358; but notice that in *Crelle*, vol. xxxv., he acknowledges his indebtedness to Dirichlet's investigations upon the class number for binary quadratics.

† We are only concerned with the positive automorphics of the ternary forms.

‡ Bachman, p. 26.

$$x' = x$$
$$y = lx - y - z$$
$$z' = px + y.$$

By writing $Z = -px + z$, T takes the form

$$x' = x$$
$$y' = \lambda x - y - z$$
$$z' = y,$$

and then T^2 is

$$x'' = x$$
$$y'' = z$$
$$z'' = \lambda x - y - z.$$

Hence if $f(x, y, z)$ denote any ternary quadratic, the most general ternary form invariant for T is

$$f(x, y, z) + f(x', y', z') + f(x'', y'', z'').$$

But this is found to be symmetrical with respect to y and z, and hence the form has an automorphic $x' = -x$, $y' = -z$, $z' = -y$, which is of the second order.

Let now T be the automorphic of the second order belonging to a ternary quadratic. Since U is also of the second order, the ternary quadratic is equivalent to a form for which T takes the form

$$x' = x$$
$$y' = lx \pm y$$
$$z' = px \pm z.$$

Since this is of the second order, we reject the positive sign, as then T is the identical operation. Putting now $kY = py - lz$, where k is the greatest common factor of l and p, T takes the* form $x' = x$, $y' = -y$, $z' = qx - z$. Finally, putting $Z = z - tx$, by properly selecting t, T takes one of the forms

$$x' = x,\ y' = -y,\ z' = -z \text{ or } x' = x,\ y' = -y,\ z' = x - z.$$

Hence if $\chi(x, y, z)$ be any definite ternary form, the most general ternary quadratic having more than one automorphic is $\chi(x, y, z) + \chi(x', y', z')$. For the first form of T this is

$$f(x, y, z) = ax^2 + by^2 + 2fyz + cz^2 = ax^2 + \phi,$$

where a, b, c, f are integers of which a is a divisor of the determinant D of $f(x, y, z)$. For the second form of T this is

$$ax^2 + by^2 + c(2z^2 - 2zx) + 2fy(2z - x),$$

* The y and z here are not the same as the previous y and z, but this need not lead to any confusion.

and by changing x into $x+2z$ and then* replacing x, y, z by y, z, x respectively, this takes the form†

$$2f(x, y, z) = a(2x+y)^2 + by^2 + 2fyz + cz^2 = a(2x+y)^2 + \phi,$$

where a, b, c, f are integers, and a is a divisor of $2D$. This proves one of Eisenstein's results.

The question now arises if any of the forms

$$ax^2 + by^2 + 2fyz + cz^2$$

corresponding to different values of a, b, c, f are equivalent, similarly for the forms

$$a(2x+y)^2 + by^2 + 2fyz + cz^2,$$

and finally if any forms of the first series are equivalent to forms of the second series. We must also find the number of automorphics of the non-equivalent forms of the series.

Suppose now that the substitution S

$$x' = l_1x + m_1y + n_1z$$
$$y' = l_2x + m_2y + n_2z$$
$$z' = l_3x + m_3y + n_3z$$

changes the form $f(x, y, z) = ax^2 + by^2 + 2fyz + cz^2 = ax^2 + \phi$ into $a_1x^2 + b_1y^2 + 2f_1yz + c_1z^2$. Writing for shortness $\dfrac{\partial f}{\partial l_1}$ instead of $\dfrac{\partial f(l_1, l_2, l_3)}{\partial l_1}$, and $\dfrac{\partial \phi}{\partial l_2}$ instead of $\dfrac{\partial \phi(l_2, l_3)}{\partial l_2}$, we find, by substituting from S and equating to zero the coefficients of xy, xz,

$$m_1\frac{\partial f}{\partial l_1} + m_2\frac{\partial f}{\partial l_2} + m_3\frac{\partial f}{\partial l_3} = 0,$$
$$n_1\frac{\partial f}{\partial l_1} + n_2\frac{\partial f}{\partial l_2} + n_3\frac{\partial f}{\partial l_3} = 0.$$

These give

$$\frac{\dfrac{\partial f}{\partial l_1}}{m_2n_3 - m_3n_2} = \frac{\dfrac{\partial f}{\partial l_2}}{m_3n_1 - m_1n_3} = \frac{\dfrac{\partial f}{\partial l_3}}{m_1n_2 - m_2n_1} = 2a_1,$$

since $$2a_1 = l_1\frac{\partial f}{\partial l_1} + l_2\frac{\partial f}{\partial l_2} + l_3\frac{\partial f}{\partial l_3}.$$

Hence $$al_1, \quad \tfrac{1}{2}\frac{\partial \phi}{\partial l_2}, \quad \tfrac{1}{2}\frac{\partial \phi}{\partial l_3}$$

* Note that if two ternary forms are improperly equivalent, they are also properly equivalent, and *vice versâ*.

† a, b, c, f are not the same as the a, b, c, f in the line above.

are all divisible by $a_1 = al_1^2 + bl_2^2 + 2fl_2l_3 + cl_3^2 = al_1^2 + \phi(l_2, l_3)$. But since ϕ is a positive definite form, a_1 is in general greater than al_1, so that if al_1 is divisible by a_1, either $l_1 = 0$ or $l_1 = \pm 1$, $\phi(l_2, l_3) = 0$. If $l_1 = 0$, $\frac{1}{2}\frac{\partial\phi}{\partial l_2}$ and $\frac{1}{2}\frac{\partial\phi}{\partial l_3}$ are both divisible by $\phi(l_2, l_3)$. This requires that ϕ should be equivalent to an ambiguous form whose middle coefficient is zero. For putting

$$\frac{\frac{\partial\phi}{\partial l_2}}{r} = \frac{\frac{\partial\phi}{\partial l_3}}{-q} = 2\phi(l_2, l_3),$$

q and r are integers satisfying the equations $l_2r - l_3q = 1$ and

$$q\frac{\partial\phi}{\partial l_2} + r\frac{\partial\phi}{\partial l_3} = 0.$$

Hence the substitution $\begin{pmatrix} l_2, & q \\ l_3, & r \end{pmatrix}$ applied to ϕ changes it into a form ϕ_1, whose middle coefficient is zero. But as $ax^2 + \phi$ is equivalent to $ax^2 + \phi_1$, if ϕ is equivalent to ϕ_1, we may take $f = 0$. In this case* the forms $ax^2 + by^2 + cz^2$ and $a_1x^2 + b_1y^2 + c_1z^2$ are equivalent if, and only if, a_1, b_1, c_1 is a permutation of a, b, c. We may note here that if $\frac{\partial\phi(l_2, l_3)}{\partial l_2}$ and $\frac{\partial\phi(l_2, l_3)}{\partial l_3}$ are both divisible by $\phi(l_2, l_3)$, then ϕ must be an ambiguous form. For putting now

$$\frac{\frac{\partial\phi}{\partial l_2}}{r} = -\frac{\frac{\partial\phi}{\partial l_3}}{q} = 2\phi(l_2, l_3),$$

the substitution $\begin{pmatrix} l_2, & q \\ l_3, & r \end{pmatrix}$, where $l_2r - l_3q = 2$, changes ϕ into a form $b_1y^2 + c_1z^2$. Hence ϕ is equivalent to a form $\frac{1}{4}(b_1y_1^2 + c_1z_1^2)$ where y_1, z_1 is a reduced substitution of determinant 2, that is

$$y_1 = y, \quad 2y, \quad 2y + z,$$
$$z_1 = 2z, \quad z, \quad z.$$

From this, obviously ϕ is an ambiguous form.

The second case $l_1 = \pm 1$, $\phi(l_2, l_3) = 0$ reduces to $l_1 = \pm 1$, $l_2 = l_3 = 0$, and then $m_1 = n_1 = 0$. Hence the forms $ax^2 + \phi$ and $a_1x^2 + \phi_1$ are equivalent only if (provided they do not come under the previous case) ϕ and ϕ_1 are equivalent, either properly or improperly. Hence if ϕ is a reduced binary form with $f \neq 0$, only the forms $ax^2 + by^2 + cz^2 \pm 2fyz$ are equivalent.

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* Clearly $l_2 = \pm 1, \ l_3 = 0,$ or $l_2 = 0 \quad l_3 = \pm 1.$

Let now $-d$ be the determinant of ϕ, so that d is a divisor of D, the determinant of the ternary forms. Call d_2 the number of classes of non-equivalent ternary forms with two automorphics when we take for ϕ the various reduced binary forms of determinant $-d$, similarly for d_4, d_6, ...

If ϕ is not an ambiguous form, it has only the two automorphics $y'=\pm y$, $z'=\pm z$, and then the ternary form has only two. Hence $2d_2=h(d)+h'(d)-A(d)-A'(d)$ where $A(d)$ and $A'(d)$ respectively are the number of properly and improperly primitive ambiguous classes of determinant $-d$.

If ϕ is an ambiguous form it admits of four automorphics (proper and improper) unless it is derived* from the forms y^2+z^2 or $2y^2+2yz+2z^2$ when it has eight or twelve automorphics, and the ternary form has the same number of automorphics as ϕ. Hence $d_4=A(d)+A'(d)$ if $d\neq 1$, 3 or D. If $d=D$, $a=1$, then the form $x^2+y^2+Dz^2$ has eight automorphics and similarly if $d=1$. If* $d=3$ the form

$$\tfrac{1}{3}Dx^2+2y^2+2yz+2z^2$$

has twelve automorphics.

Hence if $d=1$, 3 or D, $d_4=A(d)+A'(d)-1$.

Also $d_8=0$ or 1 according as d is not or is equal to D, 1.

Finally $d_{12}=0$ or 1 according as d is not or is equal to 3.

We must now carry out a similar investigation for the second group of forms which are given by

$$\begin{aligned}2f(x,y,z)&=a(2x+y)^2+by^2+2fyz+cz^2\\&=a(2x+y)^2+\phi=\chi(x,y,z),\end{aligned}$$

where c and f are even and $a\equiv b \pmod 2$. Since $a(bc-f^2)=2D$ and D has no square factors, a and b are odd. If any of these forms is equivalent to a form of the first group, by applying the substitution S and equating to zero the coefficients of xy, xz, we find that $\frac{1}{2}\frac{\partial\chi}{\partial l_1}$, $\frac{1}{2}\frac{\partial\chi}{\partial l_2}$, $\frac{1}{2}\frac{\partial\chi}{\partial l_3}$ are all divisible by $a(2l_1+l_2)^2+\phi(l_2,l_3)$. But if $2a(2l_1+l_2)$ is divisible by $2a(2l_1+l_2)^2+\phi(l_2,l_3)$, either $2l_1+l_2=\pm 1$ and $\phi(l_2,l_3)=0$ or $2l_1+l_2=0$. The first case is rejected as it reduces to $l_2=l_3=0$, $2l_1=\pm 1$. In the second case $\frac{1}{2}\frac{\partial\phi}{\partial l_2}$ and $\frac{1}{2}\frac{\partial\phi}{\partial l_3}$ are divisible by $\phi(l_2,l_3)$, and hence ϕ must be equivalent to an ambiguous form with its middle coefficient equal to zero, and then the ternary form is equivalent to a form occurring in the first series.

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* Note that we assume D has no squared factors, and also omit the trivial case $D=1$.

Now let us see if S can change χ into a form
$\chi_1 = a_1(2x+y)^2 + b_1y^2 + 2f_1yz + c_1z^2 = a_1(2x+y)^2 + \phi_1$.
The conditions for this are

$$2\left(m_1\frac{\partial\chi}{\partial l_1} + m_2\frac{\partial\chi}{\partial l_2} + m_3\frac{\partial\chi}{\partial l_3}\right) = l_1\frac{\partial\chi}{\partial l_1} + l_2\frac{\partial\chi}{\partial l_2} + l_3\frac{\partial\chi}{\partial l_3}$$

$$n_1\frac{\partial\chi}{\partial l_1} + n_2\frac{\partial\chi}{\partial l_2} + n_3\frac{\partial\chi}{\partial l_3} = 0.$$

Hence $$\frac{\partial\chi}{\partial l_1}\Big/\begin{vmatrix} 2m_2 - l_2, & 2m_3 - l_3 \\ n_2, & n_3 \end{vmatrix} = a_1,$$

so that $\dfrac{\partial\chi}{\partial l_1}, \dfrac{\partial\chi}{\partial l_2}, \dfrac{\partial\chi}{\partial l_3}$ are all divisible by $a(2l_1+l_2)^2 + \phi(l_2, l_3)$.

But if $4a(2l_1+l_2)$ is divisible by $a(2l_1+l_2)^2 + \phi(l_2, l_3)$, then $2l_1+l_2 = 0, \pm1, \pm2, \pm3$, or ±4.

If $2l_1+l_2 = \pm2$, $\phi(l_2, l_3) = 0$ or $4a$. But if $\phi(l_2, l_3) = 0$, $l_2 = l_3 = 0$, $l_1 = \pm1$, and then ϕ is equivalent (properly or improperly) to ϕ_1. In this case the forms $a(2x+y)^2 + \phi$ and $a(2x+y)^2 + \phi_1$ are equivalent. If $\phi(l_2, l_3) = 4a$, $\pm4a + \dfrac{\partial\phi}{\partial l_2}$ and $\dfrac{\partial\phi}{\partial l_3}$ are both divisible by $8a$. Hence

$$\frac{\partial\phi}{\partial l_2}, \frac{\partial\phi}{\partial l_3} \equiv 0 \ \{\mathrm{mod}\ \phi(l_2, l_3)\},$$

so that ϕ is an ambiguous form.

We note that there are no ambiguous forms of the type $(b, f, 2f)$. For as f is even, putting $f = 2k$, $4(k^2 - kb)$ must be a factor of $2D$. This is impossible as b is odd, $k^2 - k$ is even, and D has no squared factors. The only ambiguous forms arising then, are of the type $(b, 0, c)$ and the corresponding ternary form is equivalent to a form of the first series.

If $2l_1+l_2 = 0$, $\dfrac{\partial\phi}{\partial l_2}$ and $\dfrac{\partial\phi}{\partial l_3} \equiv 0 \ \{\mathrm{mod}\ \phi(l_2, l_3)\}$, and ϕ is an ambiguous form.

If $2l_1+l_2 = \pm3$, $\phi(l_2, l_3) = 3a$, and then

$$\pm6a + \frac{\partial\phi}{\partial l_2}, \frac{\partial\phi}{\partial l_3} \equiv 0 \pmod{12a}$$

Hence $$\frac{\partial\phi}{\partial l_2}, \frac{\partial\phi}{\partial l_3} \equiv 0 \ \{\mathrm{mod}\ 2\phi(l_2, l_3)\}$$

and ϕ is an ambiguous form.

If $2l_1+l_2 = \pm4$, $\phi(l_2, l_3) = 0$, and then $l_1 = \pm2$, $l_2 = l_3 = 0$, which is impossible, as the determinant of S is unity.

If $2l_1+l_2 = \pm1$, $\phi(l_2, l_3) = 0$, a or $3a$.

Clearly $\phi(l_2, l_3) = 0$ may be rejected, and $\phi(l_2, l_3) = a$ gives $\frac{\partial\phi}{\partial l_2}, \frac{\partial\phi}{\partial l_3} \equiv 0 \pmod{2a}$, so that ϕ is an ambiguous form.

Finally, suppose $\phi(l_2, l_3) = 3a$.

Here
$$\pm 2a + \frac{\partial\phi}{\partial l_2}, \frac{\partial\phi}{\partial l_3} \equiv 0 \pmod{4a},$$

or
$$\pm a + bl_2 + fl_3 \equiv 0 \pmod{2a}$$
$$fl_2 + cl_3 \equiv 0 \pmod{2a}.$$

Hence l_2 is odd, and we can write
$$bl_2 + fl_3 \equiv 0 \pmod{a},$$
$$fl_2 + cl_3 \equiv 0 \pmod{a}.$$

Since $bc - f^2$ is prime to a, l_2 and $l_3 \equiv 0 \pmod{a}$, and from $\phi(l_2, l_3) = 3a$, $3a \equiv 0 \pmod{a^2}$, so that $a = 1$ or 3. But if
$$a = 3,\ b\left(\frac{l_2}{3}\right)^2 + 2f\left(\frac{l_2}{3}\right)\left(\frac{l_3}{3}\right) + c\left(\frac{l_3}{3}\right)^2 = 1,$$
so that ϕ is an ambiguous form.

If $a = 1$, χ is equivalent to a form $(2x+y)^2 + 3y^2 + 4fyz + 2cz^2$, or more simply to the form $(2x+y)^2 + 3y^2 + 4yz + 2cz^2$. But as then $D = 3c - 2$, such a form exists only if $D + 2 \equiv 0 \pmod{3}$. Noting now that $3l_2^2 + 4l_2l_3 + 2cl_3^2 = 3$ gives* $l_2 = \pm 1$, $l_3 = 0$, we can shew that this form χ has six automorphics which are given by

$$x' = x,\ x+y,\ -x-y,\ \ y+z\ ,\ -y-z,\ -x\ ;$$
$$y' = y,\ \ -y,\ \ x-z,\ -x-y-z,\ -x+z,\ x+y+z,$$
$$z' = z,\ \ -z,\ \ z\ ,\ \ z\ \ ,\ -z\ ,\ -z.$$

Hence of the forms
$$2f(x, y, z) = a(2x+y)^2 + by^2 + 2fyz + cz^2 = a(2x+y)^2 + \phi$$
(when a is given we take for ϕ a representative from each of the classes of binary forms of determinant $-2D/a$) only the forms $a(2x+y)^2 + by^2 \pm 2fyz + cz^2$ are equivalent, provided that ϕ is not an ambiguous form. Moreover, the automorphics of $f(x, y, z)$ leave unaltered both $(2x+y)^2$ and ϕ, so that the ternary forms have two automorphics if ϕ is not an ambiguous form. An exception arises when $a = 1$, $b = 3$, and $f = 2$, and then the ternary quadratic has six automorphics.

We can now find the number of classes of ternary forms of given determinant D. First consider the case when $D = P$ is odd and without any squared factors, so that improperly

* Note that $c > 2$.

primitive forms do not arise. The forms having more than one automorphic, belong to two series of forms, of which the first is $ax^2+by^2+2fyz+cz^2$, where $d=bc-f^2$ is any divisor of P and $ad=P$. We may omit the value $d=1$, as then only the form $Px^2+y^2+z^2$ arises, but occurs again when $d=P$ as the form $x^2+y^2+Pz^2$. Call p_2 the total number of non-equivalent forms of the first series with two automorphics, and similarly for p_4, p_6, ...

Then $2p_2=\Sigma h(d)+\Sigma h_1(d)-\Sigma A(d)-\Sigma A'(d)$, where the summation refers to all the divisors d of P, unity being excluded.

$p_4=\frac{1}{3}\{\Sigma A_0(d)-1\}+\Sigma A(d)+\Sigma A'(d)-\Sigma A_0(d)+0$ or -1, according as P is not or is divisible by 3, where $A_0(d)$ stands for the number of ambiguous classes of determinant $-d$, which have their middle coefficient equal to zero. The -1 occurs with $A_0(d)$ as the form $x^2+y^2+Pz^2$ has eight automorphics, and we have divided by 3 as the form $ax^2+by^2+cz^2$ (where neither a, b nor c equals P) occurs three times, *e.g.* if $a<b<c$ it occurs as $ax^2+by^2+cz^2$, $bx^2+ay^2+cz^2$, $cx^2+ay^2+bz^2$. The 0 or -1 in p_4 is due to the fact that when $P\equiv 0 \pmod 3$, one of the ternary forms has twelve automorphics.

Lastly $p_8=1$, and $p_{12}=0$ or 1 according as P is not or is divisible by 3.

The second series of forms is given by

$$2f(x, y, z)=a(2x+y)^2+by^2+2fyz+cz^2,$$

where a is odd, $bc-f^2=2d$ and $ad=2P$. Use now q_2, q_4, ..., instead of p_2, p_4, ...

Then $q_2=\frac{1}{2}\{\Sigma h(2d)-\Sigma A(2d)\}+0$ or -1, according as $P-1$ is not or is divisible by 3, since $h'(2d)=0$. The 0 or -1 occurs because, when $P\equiv 1 \pmod 3$, one of the forms has six automorphics. Also $q_6=0$ or 1 under the same conditions. The ternary forms with $f=0$ have occurred in the first series, and must not be reckoned again. $d=1$ is included in the summation, but it may be omitted as $h(2)=A(2)$. Further, q_4, q_8, ..., are zero.

But $\sigma_2=p_2+q_2$, etc., and since the weight of the order of ternary forms* is $M=\frac{1}{24}(2P-1)$,

$$H(P)=\tfrac{1}{24}(2P-1)+\tfrac{1}{2}\sigma_2+\tfrac{3}{4}\sigma_4+\tfrac{5}{6}\sigma_4+\tfrac{7}{8}\sigma_8+\tfrac{11}{12}\sigma_{12},$$

from which

$$H(P)=\tfrac{1}{4}\{\Sigma h(d)+\Sigma h'(d)-\Sigma A(d)-\Sigma A'(d)+\Sigma h(2d)-\Sigma A(2d)\}$$
$$+\tfrac{3}{4}[\tfrac{1}{3}\Sigma A_0(d)+\Sigma A(d)+\Sigma A'(d)-\Sigma A_0(d)]+\tfrac{1}{12}(P+e),$$

* Eisenstein, *Crelle* (1851); or Bachman, p. 196.

where e depends upon the residue of $P \pmod 3$. But the part of this sum involving A is equal to zero as it reduces to $\frac{1}{4}\Sigma\{2A(d)+2A'(d)-2A_0(d)-A(2d)\}$. For suppose d has p prime factors, then $A_0(d)=2^{p-1}$, and* if $d\equiv 3 \pmod 4$, $A(d)=2^{p-1}$, $A'(d)=2^{p-1}$; but if $d\equiv 1 \pmod 4$, $A(d)=2^p$, $A'(d)=0$, $A(2d)=2^p$.

These values reduce the sum under the bracket to zero. Noting now the various cases that arise for different forms of P, we find, when $P\equiv 0 \pmod 3$,

$$e/12=-\tfrac{1}{24}-\tfrac{1}{4}-\tfrac{3}{4}+\tfrac{7}{8}+\tfrac{11}{12} \text{ and } e=9.$$

If $P\equiv +1 \pmod 3$, $e/12=-\tfrac{1}{24}-\tfrac{1}{4}+\tfrac{7}{8}+\tfrac{5}{6}-\tfrac{1}{2}$ and $e=11$.

If $P\equiv -1 \pmod 3$, $e/12=-\tfrac{1}{24}-\tfrac{1}{4}+\tfrac{7}{8}$ and $e=7$.

Hence $e=9+2\left(\frac{P}{3}\right)$ and

$$H(P)=\tfrac{1}{4}\{\Sigma h(d)+\Sigma h'(d)+\Sigma h(2d)\}+\tfrac{1}{12}\left\{P+9+2\left(\frac{P}{3}\right)\right\},$$

as given by Eisenstein.

When the determinant of the ternary forms $D=2P$, where P is odd and without squared factors, the work is more complicated. In this case both the properly primitive and the improperly primitive orders exist, and we must partition the two series of forms accordingly. Thus

$$f(x, y, z)=ax^2+by^2+cz^2+2fyz=ax^2+\phi$$

is properly primitive if either a is odd, or if a is even and ϕ is an odd form. If a is odd, let $-2d$ be the determinant of ϕ, so that d is any divisor of P unity included, and improperly primitive forms ϕ do not exist. Hence for this group of forms

$$2p_2=h(2d)-A(2d),$$

$p_4=A(2d)$ or $A(2d)-1$ according as $d\neq$ or $=P$,

$p_8=0$ or 1, according as $d\neq$ or $=P$.

If a is even, ϕ is an odd form of determinant $-d$, and

$$2p_2=h(d)-A(d),$$

$p_4=A(d)$ or $A(d)-1$, according as $d\neq$ or $=1$,

$p_8=0$ or 1, according as $d\neq$ or $=1$.

Lastly, if $f(x, y, z)$ is improperly primitive, a is even and ϕ is an even form of determinant $-d$, so that

* Mathews, *Theory of Numbers*, pp. 173, 174.

$2p_2 = h'(d) - A'(d)$,

$p_4 = A'(d)$ or $A'(d) - 1$, according as $d \neq 3$ or $d = 3$,

$p_{12} = 0$ or 1, according as $d \neq 3$ or $d = 3$.

In the second series of forms

$$2f(x, y, z) = a(2x + y)^2 + by^2 + 2fyz + cz^2 = a(2x + y)^2 + \phi,$$

since a and b are odd and f and c are even, we can denote the determinant of ϕ by $-4d$, so that $c \equiv 0 \pmod 4$. Hence $f(x, y, z)$ is properly or improperly primitive, according as $b \equiv \pm a \pmod 4$, that is, according as the generic character of ϕ

$$(-1)^{\frac{1}{2}(\phi-1)} = (-1)^{\frac{1}{2}(a \mp 1)},$$

or for half the uneven classes ϕ of determinant $-4d$. Hence for the properly primitive forms $f(x, y, z)$, remembering that the forms $a(2x + y)^2 + by^2 \pm 2fyz + cz^2$ are equivalent, and that the form

$$2x^2 + 2y^2 + \tfrac{1}{3}(D + 2)z^2 + 2xy + 2yz$$

with six automorphics belongs to the improperly primitive order

$$2p_2 = \tfrac{1}{2}h(4d) - M(4d),$$

where $M(4d)$ denotes the number of ambiguous classes of determinant $-4d$, of the type $by^2 + 4cz^2$, where $b \equiv a \pmod 4$, or since $abc = P$, $c \equiv P \pmod 4$.

Since the weight* of the properly primitive forms is $P/8$, we have for the class number

$$H(2P) = \tfrac{1}{8}P + \tfrac{1}{2}\sigma_2 + \tfrac{3}{4}\sigma_4 + \tfrac{7}{8}\sigma_8.$$

Hence

$$H(2P) = \tfrac{7}{8} + \tfrac{1}{8}P + \tfrac{1}{4}\Sigma\{h(2d) - A(2d) + h(d) - A(d)\}$$
$$+ \tfrac{1}{8}\Sigma\{h(4d) - 2M(4d)\}_{d \neq 1}$$
$$+ \tfrac{3}{4}\left[\tfrac{1}{2}\{\Sigma A_0(2d) - 1\} + \Sigma A(2d) - \Sigma A_0(2d) + \Sigma A(d) - \Sigma A_0(d)\right].$$

The -1 after $\Sigma A_0(2d)$ is due to the omission of the form $x^2 + y^2 + 2Pz^2$, which has eight automorphics, and the 2 in the denominator, to the fact that the forms $ax^2 + by^2 + 2cz^2$ occur again as $bx^2 + ay^2 + 2cz^2$ (unless $a = b \neq 1$). We note also that the forms of the second series with $f = 0$ are not considered as they occurred in the first series, hence the reason for $d \neq 1$ in the summation $\Sigma h(4d) - 2\Sigma M(4d)$ as the only form arising when $d = 1$ is $\frac{1}{2}\{P(2x + y)^2 + y^2 + 4z^2\}$.

But $h(4d) = 2h(d)$ if d is not a perfect square, and $A(2d) = A_0(2d)$, so that

$$H(2P) = \tfrac{7}{8} + \tfrac{1}{8}P + \Sigma\{\tfrac{1}{2}h(d) + \tfrac{1}{4}h(2d)\}_{d \neq 1} + \tfrac{1}{4}h(2) + \tfrac{1}{4}h(1) - \tfrac{3}{8}$$
$$+ \tfrac{1}{4}\Sigma\{\tfrac{1}{2}A(2d) + 2A(d) - 3A_0(d) - M(4d)\}_{d=1} + \tfrac{1}{4}M(4).$$

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* *Crelle*, 1851; or Bachman, p. 196.

But $M(4)=1$ or 0, according as the form $f(x, y, z)$, where

$$2f(x, y, z)=P(2x+y)^2+y^2+4z^2$$

is properly or improperly primitive, that is, according as $P\equiv\pm 1 \pmod 4$, so that

Hence
$$M(4)=\tfrac{1}{2}\left\{1+\left(\frac{-1}{P}\right)\right\}.$$

$$H(2P)=\tfrac{1}{8}\left\{P+9+\left(\frac{-1}{P}\right)\right\}+\Sigma\left\{\tfrac{1}{2}h(d)+\tfrac{1}{4}h(2d)\right\}_{d\neq 1}$$
$$+\tfrac{1}{4}\Sigma\left\{\tfrac{1}{2}A(2d)+2A(d)-3A_0(d)-M(4d)\right\}_{d=1},$$

and we shall shew that the last line is equal to $-3/8$, proving another of Eisenstein's formulæ.

Let P have p prime factors $p_1, p_2, p_3, \ldots, \equiv 1 \pmod 4$,

and q prime factors $q_1, q_2, q_3, \ldots, \equiv 3 \pmod 4$.

Let d have m prime factors of the first kind and n of the second kind, then $\frac{1}{2}A(2d)+2A(d)-3A_0(d)=0$ or 2^{m+n}, according as $d\equiv 3$ or $1 \pmod 4$. Hence

$$\Sigma\left\{\tfrac{1}{2}A(2d)+2A(d)-3A_0(d)\right\}=\Sigma_p C_m \cdot_q C_n 2^{m+n},$$

where n is even and $m\leq p$ and $n\leq q$.

This is $3^p\left\{\frac{1}{2}(1+2)^q+\frac{1}{2}(1-2)^q\right\}=\frac{1}{2}\cdot 3^p\left\{3^q+(-1)^q\right\}$. But we must subtract $\frac{3}{2}$ from this as in the calculation when $d=1$, we have reckoned $\frac{1}{2}A(2d)+2A(d)-3A_0(d)$ as $1\{m=n=0\}$ instead of its true value $-\frac{1}{2}$.

For $M(4d)$ we require the number of decompositions of P into three factors a, b and c, such that $c\equiv P \pmod 4$. Suppose that c has m_1 factors of the first kind and n_1 of the second kind, so that $n_1\equiv q \pmod 2$. Let b have m_2 factors of the first kind and n_2 of the second kind. Hence

$$M(4d)=\Sigma_p C_{m_1}\cdot_q C_{n_1}\cdot_{p-m_1} C_{m_2}\cdot_{q-n_1} C_{n_2},$$

where the summation extends to all values of m_1, n_1, m_2, n_2, satisfying $m_1+m_2\leq p$, $n_1+n_2\leq q$, $n_1\equiv q \pmod 2$. By the multinominal theorem

$$M(4d)=\tfrac{1}{2}\left\{(1+1+1)^p\pm(1-1+1)^q\right\},$$

according as q is even or odd. Hence $M(4d)=\frac{1}{2}3^p\left\{3^q+(-1)^q\right\}$, which proves the result.

By considering now the improperly primitive forms (or more simply the improperly and properly primitive forms taken together) we can prove Eisenstein's remaining result

$$H'(2P)=\tfrac{1}{4}\left\{\Sigma h(d)+\Sigma h'(d)\right\}+\tfrac{1}{24}\left\{P+6-3\left(\frac{-1}{P}\right)-4\left(\frac{P}{3}\right)\right\}.$$

ON RATIONAL APPROXIMATIONS TO $\log x$.

By *Prof. W. Burnside.*

If $P_n(x)$ is the n^{th} Legendre coefficient, and accents denote differentiation with respect to x,

$$(x^2-1)P'_{n+1}=(n+1)(xP_{n+1}-P_n),$$

and
$$-(x^2-1)P'_n=(n+1)(xP_n-P_{n+1}).$$

Hence
$$(x^2-1)(P'_nP'_{n+1}+P_{n+1}P'_n)=(n+1)(P^2_{n+1}-P^2_n),$$

or
$$\frac{d}{dx}\left\{\frac{1}{(n+1)P_nP_{n+1}}\right\}=\frac{1}{1-x^2}\left(\frac{1}{P^2_n}-\frac{1}{P^2_{n+1}}\right),$$

and
$$\frac{d}{dx}\sum_1^n\frac{1}{iP_{i-1}P_i}=\frac{1}{1-x^2}\left(1-\frac{1}{P^2_n}\right).$$

Take x real, positive and greater than unity, and integrate each side from x to $+\infty$. Then

$$\sum_1^n\frac{1}{iP_{i-1}P_i}=\tfrac{1}{2}\log\frac{x+1}{x-1}+\int_\infty^x\frac{dx}{(x^2-1)P_n^2},$$

or
$$\tfrac{1}{2}\log\frac{x+1}{x-1}=\sum_1^n\frac{1}{iP_{i-1}P_i}+\int_x^\infty\frac{dx}{(x^2-1)P_n^2}.$$

For a given value of x, greater than unity, the integral on the right approaches zero as n increases. Hence, if $x>1$,

$$\tfrac{1}{2}\log\frac{x+1}{x-1}=\sum_1^\infty\frac{1}{iP_{i-1}P_i}.$$

If
$$z=\frac{x+1}{x-1},$$

then
$$x=\frac{z+1}{z-1},$$

and when x is positive and greater than unity, so also is z. Hence the sum of n terms of the series, with $\dfrac{x+1}{x-1}$ for the argument of the Legendre coefficients, is a rational approximation to $\frac{1}{2}\log x$. The first four rational approximations that so arise are

$$\frac{x-1}{x+1},\quad\frac{3(x^2-1)}{2(x^2+4x+1)},\quad\frac{(x-1)(11x^2+38x+11)}{6(x+1)(x^2+8x+1)},$$

$$\frac{5(x^3-1)(5x^2+32x+5)}{12(x^4+16x^3+36x^2+16x+1)}.$$

The proportional error in the last for $x=4$ is one in 7000, and for $x=5$ one in 1600.

The sum of the two terms of $\Sigma \frac{1}{iP_{i-1}P_i}$, which contain P_i in the denominator is

$$\frac{1}{iP_{i-1}P_i} + \frac{1}{(i+1)P_iP_{i+1}} = \frac{(1+1)P_{i+1}+iP_{i-1}}{i(i+1)P_{i-1}P_iP_{i+1}} = \frac{(2i+1)x}{i(i+1)P_{i-1}P_{i+1}}.$$

Hence the denominator of $\sum_1^n \frac{1}{iP_{i-1}P_i}$, where brought to its lowest terms is P_n, while the degree of the numerator in x is clearly $n-1$. Hence

$$\sum_1^n \frac{1}{iP_{i-1}P_i} = \frac{f_{n-1}}{P_n},$$

where f_{n-1} is a polynomial of degree $n-1$, and

$$f_nP_n - f_{n-1}P_{n+1} = 1/(n+1).$$

Differentiating this equation, and replacing P_n' and P'_{n+1} by their above quoted values, there results

$$[(1-x^2)f_n' + (n+1)xf_n - (n+1)f_{n-1}]P_n$$
$$-[(1-x^2)f'_{n-1} - (n+1)xf_{n-1} + (n+1)f_n]P_{n+1} = 0.$$

Hence
$$(1-x^2)f_n' + (n+1)xf_n - (n+1)f_{n-1} = kP_{n+1},$$
$$(1-x^2)f'_{n-1} - (n+1)xf_{n-1} + (n+1)f_n = kP_n,$$

where k is a numerical constant. The resulting equation for f_n is found to be

$$\frac{d}{dx}\{(1-x^2)f_n'\} + (n+1)(n+2)f_n = 2kP'_{n+1},$$

and the only polynomial of degree n which satisfies this is

$$f_n = k \sum_{i=0}^{i=[\frac{1}{2}n]} \frac{2n+1-4i}{(2i+1)(n+1-i)} P_{n-2i}.$$

When this value is substituted in $f_nP_n - f_{n-1}P_{n+1} = 1/(n+1)$, it is found that k is unity. Hence

$$\sum_1^n \frac{1}{iP_{i-1}P_i} = \sum_0^{[\frac{1}{2}(n-1)]} \frac{(2n-1-4i)P_{n-1-2i}/(2i+1)(n-i)}{P_n}.$$

When here $\frac{x+1}{x-1}$ is taken as the argument of the Legendre coefficients, this expression is found to be the same as that which arises when the Gaussian method of approximate quadrature, with n ordinates, is used to calculate $\frac{1}{2}\int_1^x \frac{dx}{x}$.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

XLVII.

On Stieltjes' 'problème des moments' (continued).

1. Le Roy, in his memoir quoted in Note XLVI,* gave, as a formal solution of the *problème des moments*, the formula

$$(1.1) \qquad f(x)=\int_0^\infty J_0\{2\sqrt{(xy)}\}\sum_0^\infty \frac{(-1)^n c_n}{(n!)^2}y^n\,dy;$$

and showed that this formula really gives a solution for certain special forms of c_n.

I shall now prove

Theorem C. *If there is a solution $f(x)$ of the problem, such that*

$$(1.2) \qquad \int_0^\infty |f(x)|\,e^{k\sqrt{x}}\,dx$$

is convergent for some positive value of k, then

$$(1.3) \qquad f(x)=G\int_0^\infty J_0\{2\sqrt{(xy)}\}\,\phi(y)\,dy$$

for almost all values of x; $\phi(y)$ being the analytic function defined, for sufficiently small values of y, by the equation

$$(1.4) \qquad \phi(y)=\Sigma\frac{(-1)^n c_n}{(n!)^2}y^n,$$

and the 'generalised integral'

$$G\int_0^\infty F(y)\,dy$$

being defined as meaning

$$\lim_{\delta\to 0}\int_0^\infty e^{-\delta y}F(y)\,dy.$$

We have

$$\phi(y)=\sum_0^\infty\frac{(-1)^n c_n}{(n!)^2}y^n=\int_0^\infty f(x)\sum_0^\infty\frac{(-1)^n(xy)^n}{(n!)^2}\,dx$$

$$=\int_0^\infty f(x)\,J_0\{2\sqrt{(xy)}\}\,dx,$$

if term by term integration be legitimate. Now

* Vol. xlvi., p. 175.

$$\sum_0^\infty \frac{|s|^n}{(n!)^2} < Ke^{\sqrt{|s|}},$$

for all values of $|s|$, so that

$$\int_0^\infty |f(x)| \sum_0^\infty \frac{|xy|^n}{(n!)^2}\, dx$$

is convergent if $|y| < k$. Thus term by term integration is legitimate for sufficiently small values of $|y|$. It follows, as in § 2 of Note XLVI, that

$$\phi(y) = \int_0^\infty f(x) J_0\{2\sqrt{(xy)}\}\, dx$$

for all positive values of y, $\phi(y)$ being the analytic function defined for sufficiently small values of y by the series (1.4).

We have therefore

$$\begin{aligned}\int_0^\infty e^{-\delta y} J_0\{2\sqrt{(wy)}\}\, \phi(y)\, dy &= \int_0^\infty e^{-\delta y} J_0\{2\sqrt{(wy)}\}\, dy \int_0^\infty f(x) J_0\{2\sqrt{(xy)}\}\, dx \\ &= \int_0^\infty f(x)\, dx \int_0^\infty e^{-\delta y} J_0\{2\sqrt{(wy)}\} J_0\{2\sqrt{(xy)}\}\, dy,\end{aligned}$$

if this integral is convergent and the change of the order of integration legitimate: and this is certainly so, since the Bessel's functions are bounded and the double integral

$$\int_0^\infty |f(x)|\, dx \int_0^\infty e^{-\delta y}\, dy$$

is convergent. Now

$$\int_0^\infty e^{-\delta y} J_0\{2\sqrt{(wy)}\} J_0\{2\sqrt{(xy)}\}\, dy$$

$$= 2\int_0^\infty e^{-\delta\eta^2} J_0(2\eta\sqrt{w}) J_0(2\eta\sqrt{x})\, \eta\, d\eta = \frac{1}{\delta} e^{-(x+w)/\delta} I_0\left\{\frac{2\sqrt{(xw)}}{\delta}\right\},$$ *

where

$$I_0(t) = J_0(it),$$

so that

$$0 < I_0(t) < Ke^t \tag{1.51}$$

for all positive values of t, and

$$I_0(t) = \frac{e^t}{\sqrt{(2\pi t)}}\left\{1 + O\left(\frac{1}{t}\right)\right\} \tag{1.52}$$

when t is large.

* This formula is due to H. Weber, 'Uber einige bestimmte Integrale', *Journal für Mathematik*, vol. lxix., 1869, pp. 1–36.

2. We have thus to consider the limit, when $\delta \to 0$, of the integral

$$\mathrm{E}(\delta) = \frac{1}{\delta} \int_0^\infty f(x)\, e^{-(x+w)/\delta}\, I_0\left\{\frac{2\sqrt{(xw)}}{\delta}\right\} dx.$$

We write this integral in the form

$$\mathrm{E}(\delta) = \frac{1}{\delta}\int_0^{x_0} + \frac{1}{\delta}\int_{x_0}^{x_1} + \frac{1}{\delta}\int_{x_1}^{\infty} = \mathrm{E}_1(\delta) + \mathrm{E}_2(\delta) + \mathrm{E}_3(\delta),$$

where $0 < x_0 < w < x_1$.

In $\mathrm{E}_1(\delta)$ we use the crude inequality (1.51), which gives

$$e^{-(x+w)/\delta}\, I_0\left\{\frac{2\sqrt{(xw)}}{\delta}\right\} < Ke^{-(\sqrt{w}-\sqrt{x})^2/\delta} < Ke^{-(\sqrt{w}-\sqrt{x_0})^2/\delta},$$

so that

$$|\mathrm{E}_1(\delta)| < \frac{K}{\delta}\, e^{-(\sqrt{w}-\sqrt{x_0})^2/\delta} \int_0^{x_0} |f(x)|\, dx \to 0.$$

Similarly

$$|\mathrm{E}_3(\delta)| < \frac{K}{\delta}\, e^{-(\sqrt{x_1}-\sqrt{w})^2/\delta} \int_{x_1}^{\infty} |f(x)|\, dx \to 0.$$

We have therefore only to obtain the limit of $\mathrm{E}_2(\delta)$.

§3. Writing $x = \xi^2$, $w = \omega^2$, we have

$$\mathrm{E}_2(\delta) = \frac{2}{\delta} \int_{\sqrt{x_0}}^{\sqrt{x_1}} \xi f(\xi^2)\, e^{-(\xi^2+\omega^2)/\delta}\, I_0\left(\frac{2\xi\omega}{\delta}\right) d\xi.$$

Also, by (1.52),

$$I_0\left(\frac{2\xi\omega}{\delta}\right) = \frac{e^{2\xi\omega/\delta}}{2\sqrt{(\pi\xi\omega/\delta)}}\{1 + O(\delta)\}.$$

The term $O(\delta)$ may be neglected. For it contributes only

$$O\left\{\sqrt{\delta}\int_{\sqrt{x_0}}^{\sqrt{x_1}} \xi\, |f(\xi^2)|\, d\xi\right\} = o(1).$$

Thus we need consider only

$$I(\delta) = \frac{1}{\sqrt{(\delta\pi\omega)}} \int_{\sqrt{x_0}}^{\sqrt{x_1}} \sqrt{\xi}\, f(\xi^2)\, e^{-(\xi-\omega)^2/\delta}\, d\xi,$$

and the limit of $I(\delta)$ is $f(\omega^2) = f(w)$ for almost all values of w.*

We have therefore

$$\lim_{\delta\to 0} \int_0^\infty e^{-\delta y}\, J_0\{2\sqrt{(wy)}\}\, \phi(y)\, dy = f(w)$$

* H. Lebesgue, *loc. cit.* (Note XLVI), p. 91.

for almost all values of w; and the proof of Theorem C is completed.*

4. Theorem C asserts that the integral

$$\int_0^\infty J_0\{2\sqrt{(xy)}\}\,\phi(y)\,dy \tag{4.1}$$

is summable in a certain manner, to sum $f(x)$, for almost all values of x. Another theorem of a similar character is

THEOREM D. *Under the same conditions as those of Theorem C, the integral* (4.1) *is summable* $(C, 1)$, *to sum* $f(x)$, *for almost all values of* x; *that is to say*

$$\lim_{Y\to\infty}\frac{1}{Y}\int_0^Y (Y-y)\,J_0\{2\sqrt{(xy)}\}\,\phi(y)\,dy = f(x),$$

for almost all values of x.

The proof of this theorem is much like several of the other proofs of this note and Note XLVI; and I therefore omit it.

5. If, as in Theorem B, we suppose $f(x)$ to be subject to some more stringent condition, we can obtain alternative forms of the solution.

Suppose in particular that

$$\int_0^\infty |f(x)|\,e^{kx}\,dx \quad (k>0)$$

is convergent. Then

$$\phi(s) = \Sigma\frac{(-1)^n c_n s^n}{n!} = \int_0^\infty f(x)\,\Sigma\frac{(-1)^n (sx)^n}{n!}\,dx = \int_0^\infty f(x)\,e^{-sx}\,dx,$$

if term by term integration is legitimate, which is certainly the case if $|s|<k$; and $\phi(s)$ is plainly an analytic function of s, regular for all values of s whose real part is greater than $-k$.

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* It was proved by Sommerfeld ('Die willkürlichen Functionen in der mathematischen Physik', *Dissertation*, Königsberg, 1891) that

$$\lim_{\delta\to 0}\int_0^\infty e^{-(\delta y)^2} J_n(xy)\,y\,dy\int_\beta^\gamma J_n(y\alpha)\,\alpha f(\alpha)\,d\alpha = f(x)$$

at any point of continuity of $f(x)$ which falls inside (β, γ). The result proved here is more special in that $n=0$, but more general in other respects.

Further results concerning this integral, and the corresponding integral with $e^{-\delta y}$, will be found in Sommerfeld's dissertation and in my paper 'Further researches in the theory of divergent series and integrals' (*Trans. Camb. Phil. Soc*, vol. xxi., 1908, pp. 1–48). I take this opportunity of pointing out that a number of the formulæ of the latter paper (§§ 31–34) are incorrect owing to the omission of the factor α before $f(\alpha)$: see (*e.g.*) equation (8), p. 43. The error does not affect the validity of the arguments used, but the results are stated inaccurately, and a number of small corrections are required.

Let us write, as usual, $s=\sigma+it$, and suppose that $\kappa>-k$ and $c>0$. Then

$$\int_{\kappa-i\infty}^{\kappa+i\infty} e^{\delta s^2+cs}\,\phi(s)\,ds=\int_{\kappa-i\infty}^{\kappa+i\infty} e^{\delta s^2+cs}\,ds\int_0^\infty f(x)e^{-sx}\,dx$$

$$=\int_0^\infty f(x)\,dx\int_{\kappa-i\infty}^{\kappa+i\infty} e^{\delta s^2+(c-x)s}\,ds,$$

the convergence and equality of all the integrals which occur being secured by the convergence of the integral

$$\int_0^\infty |f(x)|\,e^{-\kappa x}\,dx\int_{-\infty}^\infty e^{-\delta t^2}\,dt.$$

But

$$\int_{\kappa-i\infty}^{\kappa+i\infty} e^{\delta s^2+(c-x)s}\,ds=2i\int_0^\infty e^{-\delta t^2}\cos(x-c)t\,dt=i\sqrt{\left(\frac{\pi}{\delta}\right)}\,e^{-(x-c)^2/4\delta}.$$

Hence

$$\int_{\kappa-i\infty}^{\kappa+i\infty} e^{\delta s^2+cs}\,\phi(s)\,ds=i\sqrt{\left(\frac{\pi}{\delta}\right)}\int_0^\infty f(x)\,e^{-(x-c)^2/4\delta}\,dx;$$

and so*

$$\lim_{\delta\to 0}\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} e^{\delta s^2+cs}\,\phi(s)\,ds=f(c)$$

for almost all values of c.

We have thus proved

THEOREM E. *If there is a solution of the problem for which*

$$\int_0^\infty |f(x)|\,e^{kx}\,dx\quad(k>0)$$

is convergent, it is given, for almost all values of x, by the formula

$$f(x)=\frac{1}{2\pi i}\,G\int_{\kappa-i\infty}^{\kappa+i\infty}\phi(s)\,e^{xs}\,ds,$$

where $\kappa>-k$, $\phi(s)$ is the analytic function defined, for sufficiently small values of s, by the series

$$\Sigma\frac{(-1)^n c_n}{n!}\,s^n,$$

* G. H. Hardy, 'Notes on some points in the Integral Calculus', No. XLII (*On Weierstrass's singular integral*), *Messenger of Mathematics*, vol. xlvi., pp. 43–48. See also Lebesgue, *loc. cit.*

and the generalised integral

$$G\int_{-\infty}^{\infty}\psi(t)\,dt$$

is defined as meaning

$$\lim_{\delta\to 0}\int_{-\infty}^{\infty}e^{-\delta t^2}\psi(t)\,dt.$$

We have also

THEOREM F. *The integral*

$$\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}\phi(s)\,e^{xs}\,ds,$$

is summable $(C,1)$, *with sum* $f(x)$, *for almost all values of* x; *that is to say*

$$\frac{1}{4\pi i T}\int_{\kappa-iT}^{\kappa+iT}(T-|t|)\,\phi(s)\,e^{xs}\,ds\to f(x),$$

as $T\to\infty$, *for almost all values of* x.

6. In Theorems C—F we have proved that, in certain circumstances, an integral is summable to sum $f(x)$. In order to prove the *convergence* of the integrals in question we must naturally assume more, as in the theory of the convergence of Fourier's series. It is not difficult to see that any of the standard hypotheses as to the behaviour of $f(x)$, in the immediate neighbourhood of the value of x considered, suffice to secure the convergence of the integral in the ordinary sense.

It is instructive to consider the application of Theorems A to F in two special cases.

(*a*) Suppose $c_n=1/(n+1)$.

If we follow the solution of Theorem A, we find

$$F(s)=\sum_0^{\infty}\frac{(-s)^n}{(n+1)\,2n!}=\frac{2}{s}(\sqrt{s}\sin\sqrt{s}+\cos\sqrt{s}-1),$$

$$\chi(u)=\frac{2}{\sqrt{u}}\int_0^{\infty}e^{-t}\sin\frac{t}{\sqrt{u}}\,\frac{dt}{t}-2\int_0^{\infty}e^{-t}\left(1-\cos\frac{t}{\sqrt{u}}\right)\frac{dt}{t^2}$$

$$=\log\left(1+\frac{1}{u}\right),$$

$$f(x)=1\quad(0<x<1),\qquad f(x)=0\quad(x>1).$$

In order to apply the method of Theorem B we take $\alpha=1$. We find

$$F(s)=\sum_0^{\infty}\frac{(-s)^n}{(n+1)!}=\frac{1-e^{-s}}{s},$$

$$\chi(u) = \int_0^\infty e^{-t}(1 - e^{-t/u})\frac{dt}{t} = \log\left(1 + \frac{1}{u}\right).$$

In the solution of Theorems C and D we have

$$\phi(y) = \sum_0^\infty \frac{(-y)^n}{n!\,(n+1)!} = \frac{J_1(2\sqrt{y})}{\sqrt{y}},$$

$$f(x) = \int_0^\infty J_0\{2\sqrt{(xy)}\}\frac{J_1(2\sqrt{y})}{\sqrt{y}}\,dy$$

$$= 2\int_0^\infty J_0(2u\sqrt{x})\,J_1(2u)\,du,$$

$$f(x) = 1 \quad (0 < x < 1), \quad f(x) = 0 \quad (x > 1).$$

Finally, in the solution of Theorems E and F, we have

$$\phi(s) = \frac{1 - e^{-s}}{s},$$

$$f(x) = \frac{1}{2\pi i}\int_{\kappa - i\infty}^{\kappa + i\infty} \frac{1 - e^{-s}}{s} e^{xs}\,ds$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin xt}{t}\,dt - \frac{1}{\pi}\int_0^\infty \frac{\sin(x-1)t}{t}\,dt,$$

which gives the same result.

(*b*) Suppose $c_n = 1$ for all values of n. Then it will be found that all our solutions fail. For example, in the last solution, we obtain

$$\phi(s) = e^{-s},$$

$$\frac{1}{2\pi i}\int_{\kappa - i\infty}^{\kappa + i\infty} e^{\delta s^2 + xs}\phi(s)\,ds = \frac{1}{2\sqrt{(\pi\delta)}}\,e^{-(x-1)^2/4\delta},$$

which tends to infinity if $x = 1$ and to zero otherwise. Thus there is no solution satisfying our conditions, as is easily proved independently.

It should be observed that, so soon as we introduce Stieltjes' generalisation of the notion of an integral, this example admits an obvious solution. For if $\psi(x) = 0$ for $0 < x < 1$, and $\psi(x) = 1$ for $x \geq 1$, we have

$$\int_0^1 x^n\,d\psi(x) = \int_0^\infty x^n\,d\psi(x) = 1$$

for all values of n.

The methods used in this note are no doubt capable of adaptation to prove the uniqueness, under appropriate conditions, of any solution by means of a Stieltjes' integral. They would however require as a preliminary the proof of certain general theorems concerning passage to a limit under the sign of Stieltjes' integration.

I observe in conclusion that, if $f(x)$ is *positive*, the condition that

$$\int_0^\infty f(x)\, e^{k\sqrt{x}}\, dx$$

is convergent for some positive k is easily shown to be equivalent to the condition that

$$c_n < 2n!\, K^n$$

for some value of K.

ON PLANE QUARTIC CURVES WITH A TACNODE

By *Prof. Harold Hilton* and *Miss D. S. Tuck.*

§ 1. As far as we know the properties of a plane quartic curve (a "4-ic") with a tacnode have never been fully discussed. A tacnode (sometimes called a "double cusp") is a double point at which two linear branches touch. Its effect on Plücker's numbers is the same as that of two nodes, if we consider two bitangents of the quartic as coinciding with the tangent at the tacnode.

In § 2 we explain our notation. In §§ 3, 4 we enumerate the principal properties of the curve with a brief indication of their proof. In § 5 we discuss the special case in which the two linear branches at the tacnode have equal and opposite curvatures. In §§ 6, 7, 8 we investigate the quartics which have another double point in addition to the tacnode. In some cases we shall give sufficient numerical data to enable the reader to draw rapidly a typical illustrative 4-ic.

§ 2. Write down the equation of a 4-ic with the point $C(0, 0, 1)$ as tacnode and $y=0$ as tacnodal tangent. Then form the polar cubic of C. It degenerates into $y=0$ and a conic; or, if the curvatures of the branches at C are equal and opposite, into $y'=0$ and a line.

Consider the former case. Take for the moment any point on the tacnodal tangent as $A(1, 0, 0)$ and the point of contact of the other tangent from A to the conic as $B(0, 1, 0)$.

Then the conic may be written $yz+x^2=0$ with a suitable choice of homogeneous coordinates, and the equation of the 4-ic takes the form

$$(yz+x^2)^2=(1-m)(x^4+p_1x^3y+p_2x^2y^2+p_3xy^3+p_4y^4)\ldots(1),$$

or, factorizing the right-hand side,

$$(yz+x^2)^2=(1-m)(x+ay)(x+by)(x+cy)(x+dy)\ldots(2).$$

So far we have supposed A any point on the tacnodal tangent. It may be readily shown that A can be chosen so that

$$p_1\equiv a+b+c+d=0 \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and we shall suppose this relation holds in future.

We shall write

$$\alpha=\tfrac{1}{2}(1+\sqrt{m}),\quad \beta=\tfrac{1}{2}(1-\sqrt{m}).$$

The tangents from the tacnode C are

$$x+ay=0,\quad x+by=0,\quad x+cy=0,\quad x+dy=0,$$

touching at

$$C_1(a,-1,a^2),\quad C_2(b,-1,b^2),\quad C_3(c,-1,c^2),\quad C_4(d,-1,d^2).$$

The line b_{12} (or b_{21})

$$\{\alpha(a+b)+\beta(c+d)\}x+(\alpha ab+\beta cd)y-z=0\ldots(4)$$

is a bitangent, touching where

$$\alpha(x+ay)(x+by)=\beta(x+cy)(x+dy)\ldots\ldots(5),$$

as may be seen by writing (4) in the form

$$yz+x^2=\alpha(x+ay)(x+by)+\beta(x+cy)(x+dy),$$

and then squaring this last equation and subtracting from (2).

Interchanging α and β we have another bitangent b_{34}.

We have thus *three pairs of bitangents* (excluding the tangent at C), b_{12} and b_{34}, b_{13} and b_{24}, b_{14} and b_{23}.

Writing now

$$v\equiv mx^2+\tfrac{1}{2}(m-1)p_1xy+yz+(m-1)ty^2,$$

$$w\equiv\{mp_2-\tfrac{1}{4}(m-1)p_1^2-2mt\}x^2+\{mp_4-(m-1)t^2\}y^2+z^2$$
$$-2tyz-p_1zx+\{mp_3-(m-1)p_1t\}xy,$$

the quartic takes the form

$$v^2+(m-1)y^2w=0 \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

We thus exhibit the 4-ic as the envelope of the family of conics $w=0$. The 4-ic touches $w=0$ at four points lying on $v=0$.

If $w=0$ is a line-pair, it is a pair of bitangents. Writing down the condition for this, we get a cubic in t whose roots are readily found to be

$$\tfrac{1}{2}(ab+cd), \quad \tfrac{1}{2}(ac+bd), \quad \tfrac{1}{2}(ad+bc).$$

To illustrate the following theorems, draw the 4-ic

$$16(3y+x^2)^2=7(x+3)(x+2)(x-1)(x-4).$$

Here A is $(\infty, 0)$, B is $(0, 0)$, C is $(0, \infty)$, C_1 is $(-3, -3)$, C_2 is $(-2, -\frac{4}{3})$, C_3 is $(1, -\frac{1}{3})$, C_4 is $(4, -1\frac{6}{3})$.

b_{14} is $3x+12y+43=0$ touching where $x=4.42, \ -3.09,$

b_{23} is $3x-12y-13=0$ ..... $-1.55, \ .22,$

b_{24} is $12x+24y+59=0$ ..... $4.59, \ -1.92,$

b_{31} is $12x-24y-29=0$ ..... $-3.32, \ .65,$

b_{34} is $15x+12y-17=0$ ..... $6.06, \ .61,$

b_{12} is $15x-12y+23=0$ ..... $-5.52, \ -1.15.$

The conics of closest contact at C are

$$24y+8x^2=\pm\sqrt{7}(2x^2-15).$$

§ 3. I. *The points of contact C_1, C_2, C_3, C_4 of the tangents from the tacnode C lie on a conic touching the 4-ic at C. Any line through C is divided harmonically by the 4-ic and conic.*

The conic is $yz+x^2=0$.

II. *C_1, C_2 and the points of contact of the bitangent b_{12} lie on a conic touching the 4-ic at C.*

On $yz+x^2=2\alpha(x+ay)(x+by)$.

III. *C_3, C_4 and the points of contact of b_{12} lie on a conic touching the 4-ic at C.*

On $yz+x^2=2\beta(x+cy)(x+dy)$.

IV. *The points of contact of a pair of bitangents b_{12}, b_{34} lie on a conic touching the 4-ic at C and dividing AB harmonically.*

On $yz+mx^2+\frac{1}{2}(m-1)(ab+cd)y^2=0$.
It is $v=0$ with $t=\frac{1}{2}(ab+cd)$.

V. *The intersections of each pair of bitangents lie on the same conic passing through A and touching BC at C.*

The locus of the pole of $y=0$ with respect to $w=0$ is (taking $p_1=0$) $p_2y^2-4zx+2p_2xy=0$.

VI. *Of the vertices of the quadrilateral formed by b_{12}, b_{34} and b_{14}, b_{23} two are the intersections of C_1C_2 and C_3C_4, C_1C_4 and C_2C_3. The other four lie on a conic touching the 4-ic at C and dividing AB harmonically.*

On $4yz + 4x^2 = (1-m)\{2x + (a+c)y\}\{2x + (b+d)y\}$.

VII. *Any pair of bitangents divides CA harmonically.*

This is true of any conic $w = 0$.

VIII. *Given any two points U, V dividing CA harmonically; the intersections of any one of the conics $v = 0$ with any one of the conics $w = 0$ lie on a conic through U and V; and all such conics touch at U and V.*

Denote by v_1, w_1 or v_2, w_2 the result of replacing t by t_1 or t_2 in v, w. Then with $p_1 = 0$ we have

$$2(t_1 - t)v_2 + w_1 = w + (m-1)(t-t_1)(t+t_1-2t_2)y^2.$$

Now choose t so as to make $w = 0$ pass through U and V.

Putting $t_1 = t_2$ we have

IX. *Given any two points U, V dividing CA harmonically; the points of contact with the 4-ic of any one of the conics $w = 0$ lie on a conic through U and V; and all such conics touch at U and V.*

Taking $t_1 = t_2 = \frac{1}{2}(ab + cd)$ and $t = \frac{1}{4}(a+c)(b+d)$ we have

X. *The points of contact of any two pairs of bitangents lie on a conic dividing CA harmonically.*

XI. *The bitangent b_{13} meets b_{12} and b_{34}, b_{14} and b_{23} in an involution with a double point on CA. The points of contact of b_{13} are a pair in the involution.*

XII. *The two bitangents b_{13} and b_{24} meet the other two pairs of bitangents in ranges whose cross-ratios have the ratio $(\beta/\alpha)^4$.*

XIII. *The conics of closest contact at the tacnode C meet the 4-ic again in two points collinear with C. They divide AB harmonically and meet on the conic of Theorem I. One of their common chords goes through A.*

They are $yz + x^2 = \pm\sqrt{(1-m)}(x^2 + \frac{1}{2}p_2y^2)$.

XIV. *There are three points $P(1, 0, -2k)$ on CA such that four of the points of contact of tangents from P lie on a line $x = ky$ through C (two at C) and four on a conic touching CA at C. They are given by*

$$\frac{\partial}{\partial k}(k^4 + p_1k^3 + p_2k^2 + p_3k + p_4) = 0.$$

§ 4. We now classify the different kinds of real tacnodal quartic.

		BITANGENTS.		
		I $m>1$	II $1>m>0$	III $0>m$
Tangents from tacnode	(i) all real	all real	all real	none real
	(ii) two real	two real	two real	none real
	(iii) none real	——	two real	four real

If we suppose the tacnodal tangent at infinity, the shape of the various types is roughly as follows:

I (i) is a pair of ovals. I (ii) is a single oval. II (i) is the type given at the end of § 2. II (ii) is similar, but without the oval. II (iii) is rather like a pair of coaxial parabolas with concavities in the same sense.* III (i) is like an hour-glass with an oval inside the waist. III (ii) is similar, but without the oval. III (iii) is rather like a pair of coaxial parabolas with concavities in opposite senses.

In no case can more than 4 of the 12 inflexions be real.

§ 5. Suppose now that the curvatures at the tacnode are equal and opposite. The points of contact of tangents from C lie on a line. Taking this line as $z=0$, we may put the equation of the 4-ic in the form

$$My^2z^2+(x^4+p_1x^3y+p_2x^2y^2+p_3xy^3+p_4y^4) \quad \ldots\ldots(1)$$

or

$$My^2z^2+(x+ay)(x+by)(x+cy)(x+dy) \quad \ldots\ldots(2),$$

where we may suppose, without loss of generality, that

$$p_1\equiv a+b+c+d=0 \quad \ldots\ldots\ldots\ldots(3).$$

The bitangent b_{12} is now

$$2\sqrt{M}z=(a+b-c-d)x+(ab-cd)y \quad \ldots\ldots(4),$$

touching where

$$(x+ay)(x+by)+(x+cy)(x+dy)=0 \quad \ldots\ldots(5).$$

It is divided harmonically by CA, CB and its points of contact.

* But the outer branch may have real inflexions and be touched by a bitangent in real points. The two branches only intersect at the tacnode. Similarly for III (iii).

Writing

$$v \equiv x^2 + \tfrac{1}{2}p_1 xy + ty^2,$$

$$w \equiv (p_2 - \tfrac{1}{4}p_1^2 - 2t)\,x^2 + (p_4 - t^2)\,y^2 + Mz^2 + (p_3 - p_1 t)\,xy,$$

the 4-ic takes the form

$$v^2 + y^3 w = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and is exhibited as the envelope of a family of conics $w = 0$ having C as the pole of AB.

Writing

$$u \equiv (p_2 + 3p_1 t + 6t^2)\,x^2 + (p_4 + p_1 t^3 + 3t^4)\,y^2 + Mz^2 + (p_3 - 3p_1 t^2 - 8t^3)\,xy,$$

the 4-ic takes the form

$$(x - ty)^3 \{x + (3t + p_1)\,y\} + y^2 u = 0 \quad \ldots\ldots\ldots (7).$$

Hence $u = 0$ is a family of conics each of which osculates the 4-ic at two points P collinear with C and meets the 4-ic again at two points Q collinear with C, such that CP and CQ trace out homographic pencils.

Choosing t so that $u = 0$ is a line-pair, we get six pairs of inflexional tangents.

We leave to the reader the investigation of the properties of the 4-ic (i). They are for the most part very similar to those of the 4-ic of §§ 2, 3, 4, or follow from the fact that the curve can be projected into the *symmetrical* curve whose Cartesian equation is obtained by writing 1 for z in (i).

As an illustrative example take

$$9y^2 + (x + 1)(x + 2)(2x + 5)(2x - 11).$$

It has the tacnode $(0, \infty)$ and bitangents

$$\pm 4y = 8x + 21 \text{ touching where } x = \pm 2.41,$$

$$\pm 2y = 6x + 7 \quad \ldots\ldots\ldots\ldots \quad \pm .5,$$

$$\pm 6y = 14x + 27 \quad \ldots\ldots\ldots\ldots \quad \pm 2.06.$$

As another example, take

$$4y^2 = 4x^3 (x + 1) + 1.$$

§ 6. If two of the tangents from C to the 4-ic tend to coincide, the 4-ic becomes unicursal and has a node or cusp B other than the tacnode C. The two tangents ultimately coincide with CB.

If CA is the tangent at C, and BA is the harmonic conjugate of BC with respect to the tangents at B, the equation of the 4-ic of § 2 takes the form

$$(yz+x^2)^2=(1-m)\,x^2\,(x^2+p_1xy+p_2y^2) \quad \ldots\ldots(1)$$

or
$$(yz+x^2)^2=(1-m)\,x^2\,(x+ay)(x+by) \quad \ldots\ldots(2).$$

The bitangents are b_1 and b_2, b_1 being

$$z=\alpha\,(p_1x+p_2y),$$

touching where

$$\alpha\,(x^2+p_1xy+p_2y^2)=\beta x^2.$$

The tangents from B are

$$z=(a\alpha+b\beta)\,x \quad \text{and} \quad z=(a\beta+b\alpha)\,x,$$

touching at the points

$$B_1(a\alpha-b\beta,\ \beta-\alpha,\ a^2\alpha^2-b^2\beta^2)$$

and $B_2(a\beta-b\alpha,\ \alpha-\beta,\ a^2\beta^2-b^2\alpha^2)$.

The tangents from C are

$$x+ay=0 \quad \text{and} \quad x+by=0$$

touching at

$$C_1(a,-1,a^2) \text{ and } C_2(b,-1,b^2).$$

B cannot be a biflecnode. It is a flecnode (one tangent at B is inflexional), if $(1-m)\,p_1^2=4p_2$. It is a cusp, if $p_2=0$.

We may exhibit the 4-ic as an envelope of conics $w=0$ by taking c, d, p_3, p_4 zero ($p_1\neq 0$) in § 2 (6). If $t=\frac{1}{2}ab$, the conic $w=0$ is the pair of bitangents. If $t=0$, the conic is the tangents from the node.

The shape of the 4-ic may be derived from § 4 by making two of the points C_1, C_2, C_3, C_4 approach coincidence (three points, if B is a cusp).

As an illustrative example, take

$$(xy+4)^2=2\,(2y+3)(y+2).$$

The tangents from the tacnode (parallel to $y=0$) touch at $(2,-2)$ and $(-\frac{8}{3},-\frac{3}{2})$. The tangents from the node (parallel to $x=0$) touch at $(\frac{5}{2},-\frac{4}{3})$ and $(1\frac{3}{6},-1\frac{2}{5})$.

The asymptotes are $x=2$ and $x=-2$ meeting the curve again at $(2,-2)$* and $(-2,-1\frac{2}{5})$. The bitangents are

$$2x=2y+7 \text{ touching where } y=-\ .72,\ -2.78,$$

$$6x=2y+7 \ \ldots\ldots\ldots\ldots\ \ldots\ldots \qquad 1.26,\ -4.76.$$

For a 4-ic with tacnode and cusp we may take

$$2y^2+x^3y\,(x+1)=x^4 \quad \text{or} \quad (xy+6)^2=9\,(2y+3).$$

* C_1 lies on a tangent at B, if $\alpha=(1-m)\,b$, which is satisfied in this case. $y=0$ is also an asymptote, being the tangent at the tacnode $(\infty, 0)$.

§ 7. I. *C_1, C_2 and the points of contact of the bitangent b_1 lie on a conic touching the 4-ic at C.*

On $yz + x^2 = 2\alpha(x^2 + p_1xy + p_2y^2)$.

II. *The points of contact of the two bitangents lie on a conic touching the 4-ic at C.*

On $2yz + 2x^2 = (1-m)(2x^2 + p_1xy + p_2y^2)$.

III. *The points of contact of the bitangents lie on a conic with B as pole of AC.*

On $4(z^2 - p_1xz + p_2x^2) = (m-1)\{(p_1^2 - 4p_2)x^2 - p_2^2y^2\}$.

IV. *The points of contact of the two bitangents and the intersections of AB with the 4-ic subtend an involution pencil at C of which CB is a double ray.*

V. *AB, C_1C_2, b_1, b_2 are all concurrent.*

VI. *A conic touches the 4-ic at C_1, C_2 and touches BB_1, BB_2.*

$$(p_1x + p_2y - z)^2 = (1-m)p_2(x^2 + p_1xy + p_2y^2).$$

VII. *A conic touches the 4-ic at C, passes through C_1 and C_2, and through the intersections of AB with the 4-ic.*

$$yz + mx^2 + (m-1)p_1xy + (m-1)p_2y^2 = 0.$$

Writing down the polar cubic of A, we have

VIII. *The points of contact of the tangents from A lie on a conic touching the 4-ic at C.*

IX. *The conics* (1) *touching the 4-ic at C and passing through B, B_1, B_2;* (2) *touching the 4-ic at C and passing through B and the other intersections E, F of the 4-ic with the tangents at B, have double contact at B and C.*

(1) is $2yz + (m-1)p_1xy + 2mx^2 = 0$,

(2) is $2yz + (m-1)p_1xy + mx^2 = 0$.

X. *If the line EF meets the 4-ic again at H and K, the conic CB_1B_2HK touches CA, BH, BK.*

$$(m-1)\{(m-1)p_1^2 + 4p_2\}y^2 + 2m(m-1)p_1xy = 4m(yz + x^2).$$

XI. *The points of contact of b_1 and its intersections with BB_1, BB_2 form an involution with a double point on CA.*

XII. *The points B, B_1 and the intersections of BB_1 with the bitangents form an involution with a double point on CA.*

XIII. *If U and V divide CA harmonically, a conic through the points of contact of the bitangents passes through U, V and*

touches the conic BB_1B_2UV at U and V.

See Theorem VIII of § 3.

XIV. *The six inflexions of a unicursal tacnodal 4-ic lie on a conic.*

The method of Richmond and Stuart in *Proc. London Math. Soc.*, II. 1 (1903), p. 130, gives the equation of this conic, if we use the quadratic transformation which consists in replacing y by x^2/y.* The equation is

$$(4mp_2 - 3mp_1^2 - 6p_1^2)\,x^2 - 2p_2\,(4p_2 + mp_1^2 - p_1^2)\,y^2 - 8z^2$$
$$+ (20p_2 - p_1^2)\,yz - \tfrac{1}{2}p_1\,(4mp_2 + 28p_2 + 3mp_1^2 - 3p_1^2)\,xy + 16p_1xz = 0.$$

We suppress the details of the investigation.

§ 8. If the curvatures of the branches at the tacnode of a unicursal 4-ic are equal and opposite, its equation may be written

$$My^2z^2 + x^2\,(x^2 + p_1xy + p_2y^2) = 0$$

or
$$My^2z^2 + x^2\,(x + ay)(x + by) = 0.$$

The bitangents are

$$\pm 2\sqrt{Mz} = p_1x + p_2y$$

touching where $2x^2 + p_1xy + p_2y^2 = 0$.

The tangents from the node are

$$\pm 2\sqrt{Mz} = (a - b)\,x$$

touching where $2x + p_1y = 0$.

It cannot have a flecnode, but has a biflecnode (each tangent at B inflexional) if $p_1 = 0$, and has a cusp if $p_2 = 0$.

Its properties may be obtained by slight modifications of those in § 7, or by putting c, d, p_3, p_4 zero in § 5, or from the fact that the curve may be projected into the symmetrical curve got by replacing z by 1 in its equation. We leave this to the reader.

As illustrative examples we may take $x^2y^2 = 4\,(y+4)\,(y-1)$ with bitangents $\pm 2x = y + 3$, and $y^2\,(x + y)^2 = x^2\,(y + 2)$; or for quartics with tacnode and cusp

$$x^2y^2 = 16\,(y - 1) \quad \text{and} \quad x^2y^2 = 4\,(y + 1).$$

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* Instead of replacing x, y, z by their reciprocals.

In discussing this inflexions-conic and other properties of unicursal 4-ics we have received valuable help from Mr. A. E. Jolliffe.

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TABLE OF BINOMIAL-THEOREM COEFFICIENTS

By *J. W. L. Glaisher.*

§1. At various times I have required the numerical values of some of the binomial-theorem coefficients, *i.e.* the values of the coefficient of x^r in the expansion of $(1+x)^n$, for comparatively large values of the integers n and r. If obtained separately such coefficients need a certain amount of calculation, and there is no obvious method of verifying the results: but if the whole system of coefficients is calculated continuously for consecutive values of n they can be obtained with great ease by mere addition, and admit of ready and complete verification. I have therefore thought it worth while to publish the present table of all the coefficients for the first 50 values of n. I have had the table by me for a long time, and have found it to be useful.

§ 2. Denoting by $(n)_r$ the coefficient of x^r in the expansion of $(1+x)^n$, so that

$$(1+x)^n = 1 + (n)_1 x + (n)_2 x^2 + \ldots + (n)_{n-1} x^{n-1} + x^n;$$

then
$$(n)_r = \frac{n\,(n-1)\,(n-2)\ldots(n-r+1)}{r\,!},$$
$$= \frac{n\,!}{r\,!\,(n-r)\,!},$$

and
$$(n)_r = (n)_{n-r}, \quad (n)_0 = (n)_n = 1.$$

The coefficient $(n)_r$ also expresses the number of combinations of n things r together. It occurs as a fundamental quantity in many investigations.*

§ 3. The coefficients were obtained by simple addition from the formula

$$(n)_r = (n-1)_r + (n-1)_{r-1},$$

and the set of coefficients for each value of n was verified by the formula

$$(n)_0 + (n)_1 + (n)_2 + \ldots + (n)_{n-1} + (n)_n = 2^n.$$

§ 4. In the table each set of coefficients belonging to the same value of n is printed as a separate group, the value of n

* In connection with binomial-theorem coefficients I may refer to three papers of my own relating to their residues when the modulus is a prime or (in certain cases) a power of a prime, which were published in vol. xxx. (1899) of the *Quarterly Journal* (pp. 150-156, 349-360, 361-383). The formulæ in these papers afford convenient verifications of the value of any given value of $(n)_r$.

being the heading of the group. The number in brackets is the value of r, and the number immediately following it is the value of $(n)_r$. Thus for example the value of $(15)_5$, that is, the coefficient of x^5 in the expansion of $(1+x)^{15}$, is 3 003, the value of $(29)_{15}$ is 77 558 760, &c.

2

(0) 1, (1) 2, (2) 1.

3

(0) 1, (1) 3, (2) 3, (3) 1.

4

(0) 1, (1) 4, (2) 6, (3) 4, (4) 1.

5

(0) 1, (1) 5, (2) 10, (3) 10, (4) 5, (5) 1.

6

(0) 1, (1) 6, (2) 15, (3) 20, (4) 15, (5) 6, (6) 1.

7

(0) 1, (1) 7, (2) 21, (3) 35, (4) 35, (5) 21, (6) 7, (7) 1.

8

(0) 1, (1) 8, (2) 28, (3) 56, (4) 70, (5) 56, (6) 28, (7) 8, (8) 1.

9

(0) 1, (1) 9, (2) 36, (3) 84, (4) 126, (5) 126, (6) 84, (7) 36, (8) 9, (9) 1.

10

(0) 1, (1) 10, (2) 45, (3) 120, (4) 210, (5) 252, (6) 210, (7) 120, (8) 45, (9) 10, (10) 1.

11

(0) 1, (1) 11, (2) 55, (3) 165, (4) 330, (5) 462, (6) 462, (7) 330, (8) 165, (9) 55, (10) 11, (11) 1.

12

(0) 1, (1) 12, (2) 66, (3) 220, (4) 495, (5) 792, (6) 924, (7) 792, (8) 495, (9) 220, (10) 66, (11) 12, (12) 1.

13

(0) 1, (1) 13, (2) 78, (3) 286, (4) 715, (5) 1 287, (6) 1 716, (7) 1 716, (8) 1 287, (9) 715, (10) 286, (11) 78, (12) 13, (13) 1.

14

(0) 1, (1) 14, (2) 91, (3) 364, (4) 1 001, (5) 2 002, (6) 3 003, (7) 3 432, (8) 3 003, (9) 2 002, (10) 1 001, (11) 364, (12) 91, (13) 14, (14) 1.

15

(0) 1, (1) 15, (2) 105, (3) 455, (4) 1 365, (5) 3 003, (6) 5 005, (7) 6 435, (8) 6 435, (9) 5 005, (10) 3 003, (11) 1 365, (12) 455, (13) 105, (14) 15, (15) 1.

16

(0) 1, (1) 16, (2) 120, (3) 560, (4) 1 820, (5) 4 368, (6) 8 008, (7) 11 440, (8) 12 870, (9) 11 440, (10) 8 008, (11) 4 368, (12) 1 820, (13) 560, (14) 120, (15) 16, (16) 1.

17

(0) 1, (1) 17, (2) 136, (3) 680, (4) 2 380, (5) 6 188, (6) 12 376, (7) 19 448, (8) 24 310, (9) 24 310, (10) 19 448, (11) 12 376, (12) 6 188, (13) 2 380, (14) 680, (15) 136, (16) 17, (17) 1.

18

(0) 1, (1) 18, (2) 153, (3) 816, (4) 3 060, (5) 8 568, (6) 18 564, (7) 31 824, (8) 43 758, (9) 48 620, (10) 43 758, (11) 31 824, (12) 18 564, (13) 8 568, (14) 3 060, (15) 816, (16) 153, (17) 18, (18) 1.

19

(0) 1, (1) 19, (2) 171, (3) 969, (4) 3 876, (5) 11 628, (6) 27 132, (7) 50 388, (8) 75 582, (9) 92 378, (10) 92 378, (11) 75 582, (12) 50 388, (13) 27 132, (14) 11 628, (15) 3 876, (16) 969, (17) 171, (18) 19, (19) 1.

20

(0) 1, (1) 20, (2) 190, (3) 1 140, (4) 4 845, (5) 15 504, (6) 38 760, (7) 77 520, (8) 125 970, (9) 167 960, (10) 184 756, (11) 167 960, (12) 125 970, (13) 77 520, (14) 38 760, (15) 15 504, (16) 4 845, (17) 1 140, (18) 190, (19) 20, (20) 1.

21

(0) 1, (1) 21, (2) 210, (3) 1 330, (4) 5 985, (5) 20 349, (6) 54 264, (7) 116 280, (8) 203 490, (9) 293 930, (10) 352 716, (11) 352 716, (12) 293 930, (13) 203 490, (14) 116 280, (15) 54 264, (16) 20 349, (17) 5 985, (18) 1 330, (19) 210, (20) 21, (21) 1.

22

(0) 1, (1) 22, (2) 231, (3) 1 540, (4) 7 315, (5) 26 334, (6) 74 613, (7) 170 544, (8) 319 770, (9) 497 420, (10) 646 646, (11) 705 432, (12) 646 646, (13) 497 420, (14) 319 770, (15) 170 544, (16) 74 613, (17) 26 334, (18) 7 315, (19) 1 540, (20) 231, (21) 22, (22) 1.

23

(0) 1, (1) 23, (2) 253, (3) 1 771, (4) 8 855, (5) 33 649, (6) 100 947, (7) 245 157, (8) 490 314, (9) 817 190, (10) 1 144 066, (11) 1 352 078, (12) 1 352 078, (13) 1 144 066, (14) 817 190, (15) 490 314, (16) 245 157, (17) 100 947, (18) 33 649, (19) 8 855, (20) 1 771, (21) 253, (22) 23, (23) 1.

24

(0) 1, (1) 24, (2) 276, (3) 2 024, (4) 10 626, (5) 42 504,
(6) 134 596, (7) 346 104, (8) 735 471, (9) 1 307 504, (10) 1 961 256,
(11) 2 496 144, (12) 2 704 156, (13) 2 496 144, (14) 1 961 256,
(15) 1 307 504, (16) 735 471, (17) 346 104, (18) 134 596,
(19) 42 504, (20) 10 626, (21) 2 024, (22) 276, (23) 24, (24) 1.

25

(0) 1, (1) 25, (2) 300, (3) 2 300, (4) 12 650, (5) 53 130,
(6) 177 100, (7) 480 700, (8) 1 081 575, (9) 2 042 975,
(10) 3 268 760, (11) 4 457 400, (12) 5 200 300, (13) 5 200 300,
(14) 4 457 400, (15) 3 268 760, (16) 2 042 975, (17) 1 081 575,
(18) 480 700, (19) 177 100, (20) 53 130, (21) 12 650, (22) 2 300,
(23) 300, (24) 25, (25) 1.

26

(0) 1, (1) 26, (2) 325, (3) 2 600, (4) 14 950, (5) 65 780,
(6) 230 230, (7) 657 800, (8) 1 562 275, (9) 3 124 550,
(10) 5 311 735, (11) 7 726 160, (12) 9 657 700, (13) 10 400 600,
(14) 9 657 700, (15) 7 726 160, (16) 5 311 735, (17) 3 124 550,
(18) 1 562 275, (19) 657 800, (20) 230 230, (21) 65 780,
(22) 14 950, (23) 2 600, (24) 325, (25) 26, (26) 1.

27

(0) 1, (1) 27, (2) 351, (3) 2 925, (4) 17 550, (5) 80 730,
(6) 296 010, (7) 888 030, (8) 2 220 075, (9) 4 686 825,
(10) 8 436 285, (11) 13 037 895, (12) 17 383 860, (13) 20 058 300,
(14) 20 058 300, (15) 17 383 860, (16) 13 037 895, (17) 8 436 285,
(18) 4 686 825, (19) 2 220 075, (20) 888 030, (21) 296 010,
(22) 80 730, (23) 17 550, (24) 2 925, (25) 351, (26) 27, (27) 1.

28

(0) 1, (1) 28, (2) 378, (3) 3 276, (4) 20 475, (5) 98 280,
(6) 376 740, (7) 1 184 040, (8) 3 108 105, (9) 6 906 900,
(10) 13 123 110, (11) 21 474 180, (12) 30 421 755, (13) 37 442 160,
(14) 40 116 600, (15) 37 442 160, (16) 30 421 755, (17) 21 474 180,
(18) 13 123 110, (19) 6 906 900, (20) 3 108 105, (21) 1 184 040,
(22) 376 740, (23) 98 280, (24) 20 475, (25) 3 276, (26) 378,
(27) 28, (28) 1.

29

(0) 1, (1) 29, (2) 406, (3) 3 654, (4) 23 751, (5) 118 755,
(6) 475 020, (7) 1 560 780, (8) 4 292 145, (9) 10 015 005,
(10) 20 030 010, (11) 34 597 290, (12) 51 895 935, (13) 67 863 915,
(14) 77 558 760, (15) 77 558 760, (16) 67 863 915, (17) 51 895 935,
(18) 34 597 290, (19) 20 030 010, (20) 10 015 005, (21) 4 292 145,
(22) 1 560 780, (23) 475 020, (24) 118 755, (25) 23 751, (26) 3 654,
(27) 406, (28) 29, (29) 1.

30

(0) 1, (1) 30, (2) 435, (3) 4 060, (4) 27 405, (5) 142 506,
(6) 593 775, (7) 2 035 800, (8) 5 852 925, (9) 14 307 150,
(10) 30 045 015, (11) 54 627 300, (12) 86 493 225, (13) 119 759 850,
(14) 145 422 675, (15) 155 117 520, (16) 145 422 675,
(17) 119 759 850, (18) 86 493 225, (19) 54 627 300, (20) 30 045 015,
(21) 14 307 150, (22) 5 852 925, (23) 2 035 800, (24) 593 775,
(25) 142 506, (26) 27 405, (27) 4 060, (28) 435, (29) 30, (30) 1.

31

(0) 1, (1) 31, (2) 465, (3) 4 495, (4) 31 465, (5) 169 911,
(6) 736 281, (7) 2 629 575, (8) 7 888 725, (9) 20 160 075,
(10) 44 352 165, (11) 84 672 315, (12) 141 120 525,
(13) 206 253 075, (14) 265 182 525, (15) 300 540 195,
(16) 300 540 195, (17) 265 182 525, (18) 206 253 075,
(19) 141 120 525, (20) 84 672 315, (21) 44 352 165,
(22) 20 160 075, (23) 7 888 725, (24) 2 629 575, (25) 736 281,
(26) 169 911, (27) 31 465, (28) 4 495, (29) 465, (30) 31, (31) 1.

32

(0) 1, (1) 32, (2) 496, (3) 4 960, (4) 35 960, (5) 201 376,
(6) 906 192, (7) 3 365 856, (8) 10 518 300, (9) 28 048 800,
(10) 64 512 240, (11) 129 024 480, (12) 225 792 840,
(13) 347 373 600, (14) 471 435 600, (15) 565 722 720,
(16) 601 080 390, (17) 565 722 720, (18) 471 435 600,
(19) 347 373 600, (20) 225 792 840, (21) 129 024 480,
(22) 64 512 240, (23) 28 048 800, (24) 10 518 300, (25) 3 365 856,
(26) 906 192, (27) 201 376, (28) 35 960, (29) 4 960, (30) 496,
(31) 32, (32) 1.

33

(0) 1, (1) 33, (2) 528, (3) 5 456, (4) 40 920, (5) 237 336,
(6) 1 107 568, (7) 4 272 048, (8) 13 884 156, (9) 38 567 100,
(10) 92 561 040, (11) 193 536 720, (12) 354 817 320,
(13) 573 166 440, (14) 818 809 200, (15) 1 037 158 320,
(16) 1 166 803 110, (17) 1 166 803 110, (18) 1 037 158 320,
(19) 818 809 200, (20) 573 166 440, (21) 354 817 320,
(22) 193 536 720, (23) 92 561 040, (24) 38 567 100, (25) 13 884 156,
(26) 4 272 048, (27) 1 107 568, (28) 237 336, (29) 40 920,
(30) 5 456, (31) 528, (32) 33, (33) 1.

34

(0) 1, (1) 34, (2) 561, (3) 5 984, (4) 46 376, (5) 278 256,
(6) 1 344 904, (7) 5 379 616, (8) 18 156 204, (9) 52 451 256,
(10) 131 128 140, (11) 286 097 760, (12) 548 354 040,
(13) 927 983 760, (14) 1 391 975 640, (15) 1 855 967 520,
(16) 2 203 961 430, (17) 2 333 606 220, (18) 2 203 961 430,

34 (continued)

(19) 1 855 967 520, (20) 1 391 975 640, (21) 927 983 760,
(22) 548 354 040, (23) 286 097 760, (24) 131 128 140,
(25) 52 451 256, (26) 18 156 204, (27) 5 379 616, (28) 1 344 904,
(29) 278 256, (30) 46 376, (31) 5 984, (32) 561, (33) 34, (34) 1.

35

(0) 1, (1) 35, (2) 595, (3) 6 545, (4) 52 360, (5) 324 632,
(6) 1 623 160, (7) 6 724 520, (8) 23 535 820, (9) 70 607 460,
(10) 183 579 396, (11) 417 225 900, (12) 834 451 800,
(13) 1 476 337 800, (14) 2 319 959 400, (15) 3 247 943 160,
(16) 4 059 928 950, (17) 4 537 567 650, (18) 4 537 567 650,
(19) 4 059 928 950, (20) 3 247 943 160, (21) 2 319 959 400,
(22) 1 476 337 800, (23) 834 451 800, (24) 417 225 900,
(25) 183 579 396, (26) 70 607 460, (27) 23 535 820, (28) 6 724 520,
(29) 1 623 160, (30) 324 632, (31) 52 360, (32) 6 545, (33) 595,
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36

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(13) 2 310 789 600, (14) 3 796 297 200, (15) 5 567 902 560,
(16) 7 307 872 110, (17) 8 597 496 600, (18) 9 075 135 300,
(19) 8 597 496 600, (20) 7 307 872 110, (21) 5 567 902 560,
(22) 3 796 297 200, (23) 2 310 789 600, (24) 1 251 677 700,
(25) 600 805 296, (26) 254 186 856, (27) 94 143 280,
(28) 30 260 340, (29) 8 347 680, (30) 1 947 792, (31) 376 992,
(32) 58 905, (33) 7 140, (34) 630, (35) 36, (36) 1.

37

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38

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(10) 472 733 756, (11) 1 203 322 288, (12) 2 707 475 148,
(13) 5 414 950 296, (14) 9 669 554 100, (15) 15 471 286 560,
(16) 22 239 974 430, (17) 28 781 143 380, (18) 33 578 000 610,

38 (continued)

(19) 35 345 263 800, (20) 33 578 000 610, (21) 28 781 143 380,
(22) 22 239 974 430, (23) 15 471 286 560, (24) 9 669 554 100,
(25) 5 414 950 296, (26) 2 707 475 148, (27) 1 203 322 288,
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39

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(19) 68 923 264 410, (20) 68 923 264 410, (21) 62 359 143 990,
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(31) 61 523 748, (32) 15 380 937, (33) 3 262 623, (34) 575 757,
(35) 82 251, (36) 9 139, (37) 741, (38) 39, (39) 1.

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(19) 131 282 408 400, (20) 137 846 528 820, (21) 131 282 408 400,
(22) 113 380 261 800, (23) 88 732 378 800, (24) 62 852 101 650,
(25) 40 225 345 056, (26) 23 206 929 840, (27) 12 033 222 880,
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(13) 17 620 076 360, (14) 35 240 152 720, (15) 63 432 274 896,
(16) 103 077 446 706, (17) 151 584 480 450, (18) 202 112 640 600,
(19) 244 662 670 200, (20) 269 128 937 220, (21) 269 128 937 220,
(22) 244 662 670 200, (23) 202 112 640 600, (24) 151 584 480 450,
(25) 103 077 446 706, (26) 63 432 274 896, (27) 35 240 152 720,
(28) 17 620 076 360, (29) 7 898 654 920, (30) 3 159 461 968,
(31) 1 121 099 408, (32) 350 343 565, (33) 95 548 245,
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(38) 10 660, (39) 820, (40) 41, (41) 1.

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(13) 25 518 731 280, (14) 52 860 229 080, (15) 98 672 427 616,
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(22) 513 791 607 420, (23) 446 775 310 800, (24) 353 697 121 050,
(25) 254 661 927 156, (26) 166 509 721 602, (27) 98 672 427 616,
(28) 52 860 229 080, (29) 25 518 731 280, (30) 11 058 116 888,
(31) 4 280 561 376, (32) 1 471 442 973, (33) 445 891 810,
(34) 118 030 185, (35) 26 978 328, (36) 5 245 786, (37) 850 668,
(38) 111 930, (39) 11 480, (40) 861, (41) 42, (42) 1.

43

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(10) 1 917 334 783, (11) 5 752 004 349, (12) 15 338 678 264,
(13) 36 576 848 168, (14) 78 378 960 360, (15) 151 532 656 696,
(16) 265 182 149 218, (17) 421 171 648 758, (18) 608 359 048 206,
(19) 800 472 431 850, (20) 960 566 918 220, (21) 1 052 049 481 860,
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44

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(10) 2 481 256 778, (11) 7 669 339 132, (12) 21 090 682 613,
(13) 51 915 526 432, (14) 114 955 808 528, (15) 229 911 617 056,
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(21) 2 012 616 400 080, (22) 2 104 098 963 720,
(23) 2 012 616 400 080, (24) 1 761 039 350 070,
(25) 1 408 831 480 056, (26) 1 029 530 696 964, (27) 686 353 797 976,
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45 (continued)

(13) 73 006 209 045, (14) 166 871 334 960, (15) 344 867 425 584,
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(19) 2 438 362 177 020, (20) 3 169 870 830 126,
(21) 3 773 655 750 150, (22) 4 116 715 363 800,
(23) 4 116 715 363 800, (24) 3 773 655 750 150,
(25) 3 169 870 830 126, (26) 2 438 362 177 020,
(27) 1 715 884 494 940, (28) 1 103 068 603 890, (29) 646 626 422 970,
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(36) 886 163 135, (37) 215 553 195, (38) 45 379 620,
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46

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(23) 8 233 430 727 600, (24) 7 890 371 113 950,
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(35) 13 340 783 196, (36) 4 076 350 421, (37) 1 101 716 330,
(38) 260 932 815, (39) 53 524 680, (40) 9 366 819, (41) 1 370 754,
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50 (continued)

(22) 88 749 815 264 600, (23) 108 043 253 365 600,
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(26) 121 548 660 036 300, (27) 108 043 253 365 600,
(28) 88 749 815 264 600, (29) 67 327 446 062 800,
(30) 47 129 212 243 960, (31) 30 405 943 383 200,
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(34) 4 923 689 695 575, (35) 2 250 829 575 120, (36) 937 845 656 300,
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(43) 99 884 400, (44) 15 890 700, (45) 2 118 760, (46) 230 300,
(47) 19 600, (48) 1 225, (49) 50, (50) 1.

A THEOREM OF LIOUVILLE'S TYPE.

By *G. N. Watson.*

1. The theorem, due to Cauchy and Liouville,* that if $f(z)$ is analytic for all finite values of $|z|$ and is bounded when $|z| \to \infty$, then $f(z)$ is constant, is well known and is to be found in various treatises on function-theory. More recently it has been shewn† by Phragmen and Lindelöf that integral functions satisfying certain inequalities must be constants. I have applied‡ methods indicated by Phragmen's analysis in his first paper to prove the following theorem:

Let $f(z)$ be analytic at all points inside an infinite sector of the Argand diagram, whose vertex is at the origin, and let

$$|f(z)| < A \exp(-|z|)$$

(where A is a constant), at all points inside the sector. Then $f(z)$ is zero, if the angle of the sector exceeds *two right angles.*

[The theorem is obviously untrue if the angle of the sector is less than two right angles; this may be seen by taking the boundaries of the sector to be the rays $\arg z = \pm(\frac{1}{2}\pi - \delta)$ and taking $f(z)$ to be $\exp(-z \operatorname{cosec} \frac{1}{2}\delta)$].

In this paper I propose to shew that the theorem is true, *i.e.* $f(z)$ is zero, if the angle of the sector is *equal* to two right angles, so that I fill up the lacuna between the two theorems previously known.

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* See Whittaker and Watson, *Modern Analysis*, pp. 105, 424.

† Phragmen, *Acta Mathematica*, xxviii. (1904), pp. 351–368; Phragmen and Lindelöf, *ibid.*, xxxi. (1908), pp. 381–406.

‡ *Phil. Trans.*, 211 A, pp. 295–300.

It is not difficult to prove the theorem by means of a result given on p. 403 of the second paper by Phragmen-Lindelöf; but as this result is obtained by somewhat abstruse methods, I propose to give a proof by the methods which I used previously. The analysis does not require the use of any results deeper than the contour-integral formula for the n^{th} derivate of an analytic function, and standard theorems on differentiation of infinite integrals under the integral sign.

2. Let $f(\zeta)$ be analytic and subject to the inequality

$$|f(\zeta)| < A \exp(-|\zeta|)$$

for all values of ζ such that* $R(\zeta) \geq 0$; then we shall prove that $f(\zeta)$ is zero.

Put $\zeta = z+2$ and let $f(\zeta) \equiv g(z)$, so that $g(z)$ is analytic when $R(z) \geq -2$; and it is easy to see that

$$|g(z)| < Ae^3 \exp(-|z|) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The n^{th} derivate of $g(z)$ is given by the formula

$$g^{(n)}(z) = \frac{n!}{2\pi i} \int \frac{g(u)}{(u-z)^{n+1}}\, du,$$

where the contour may be taken to be a circle of unit radius whose centre is at $u = z$, provided that $R(z) \geq 0$. On the contour

$$|g(u)| < Ae^3 \exp(-|z|),$$

and so, when $R(z) \geq 0$, we have

$$|g^{(n)}(z)| < n!\, Ae^3 \exp(-|z|) \quad \ldots\ldots\ldots\ldots\ldots (2).$$

3. In order to prove that $g(z)$ is zero, we shall first shew that the auxiliary function $\mathfrak{F}(z)$, defined by the equation

$$\mathfrak{F}(z) = \int_0^\infty g(tz)\, e^{-\frac{1}{2}t}\, dt \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

is zero. In the course of this investigation we shall have to make use of the following theorem†:

If, in the integral $\Phi(z) = \int_0^\infty \phi(z, t)\, dt$, ϕ *is an analytic function of* z *at all points of a region* T, *for all values of* t *greater than or equal to zero,* $\Phi(z)$ *is also analytic at all points inside* T *if a positive function*‡ $M(t)$, *independent of* z, *exists*

* There is obviously no loss of real generality in taking the boundary of the sector to be the imaginary axis.

† Bromwich, *Infinite Series*, p. 438.

‡ We shall use M as a functional symbol in this sense throughout the paper.

such that $|(\partial\phi/\partial z)| < M(t)$ *and* $\int_0^\infty M(t)\,dt$ *is convergent.*

It simplifies the analysis to use the symbol $\int_{(\theta)}$ to denote an integral taken along that ray from the origin to infinity which is inclined at an angle θ to the real axis.

4. We shall first investigate the analytic continuation of $\mathfrak{F}(z)$. From the theorem of § 3 it follows that $\mathfrak{F}(z)$ is analytic when $R(z) > 0$; for we may take $M(t)$ to be equal to $Ae^{3}te^{-\frac{1}{2}t}$. Next, if $\frac{1}{2}\pi \geq \arg z \geq 0$ and $|z| > 0$, we have

$$\int_{(0)} g(tz)\, e^{-\frac{1}{2}t}\,dt = \int_{(-\frac{1}{2}\pi)} g(tz)\, e^{-\frac{1}{2}t}\,dt,$$

by Cauchy's theorem; for the integrand is analytic in the region between the two rays and $\int f(tz)\, e^{-\frac{1}{2}t}\,dt$, taken round an infinite quadrant between the rays, is zero on account of the inequality (1). Hence, when $\frac{1}{2}\pi \geq \arg z \geq 0$ and* $|z| \geq \delta$, we have

$$\mathfrak{F}(z) = \int_{(-\frac{1}{2}\pi)} g(tz)\, e^{-\frac{1}{2}t}\,dt \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

so that

$$\mathfrak{F}(z) = -i\int_0^\infty g(-iuz)\, e^{\frac{1}{2}iu}\,du.$$

Now the function on the right is an analytic function† of z when $0 < \arg z < \pi$ and $|z| > \delta$; and so it may be used to define the analytic continuation of $\mathfrak{F}(z)$ over this extended region.

Similarly the analytic continuation of $\mathfrak{F}(z)$ over the region $0 > \arg z > -\pi$ is given by the equation

$$\mathfrak{F}(z) = \int_{(\frac{1}{2}\pi)} g(tz)\, e^{-\frac{1}{2}t}\,dt \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Let us now find the analytic continuation of $\mathfrak{F}(z)$ in a region in which $\arg z > \pi$.

Let μ be any positive acute angle, and suppose that

$$|z| > \sin\mu, \quad |z| > \delta, \quad \mu < \arg z < \pi.$$

Then

$$\int_{(-\frac{1}{2}\pi)} g(tz)\, e^{-\frac{1}{2}t}\,dt = \int_{(-\frac{1}{2}\pi-\mu)} g(tz)\, e^{-\frac{1}{2}t}\,dt;$$

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* It is supposed that δ is an arbitrarily small positive number.

† We may take $M(u) = Ae^{3}ue^{-\delta u}$.

for, by Cauchy's theorem, the difference between the integrals is

$$\lim_{N\to\infty}\int_{-\frac{1}{2}\pi}^{-\frac{1}{2}\pi-\mu} g(tz)\,e^{-\frac{1}{2}t}\frac{dt}{d\theta}\,d\theta,$$

where $t=Ne^{i\theta}$.

Now, when $t=Ne^{i\theta}$,

$$|g(tz)|<Ae^{N}\exp(-N\sin\mu),$$

$$|e^{-\frac{1}{2}t}|\leq\exp(\tfrac{1}{2}N\sin\mu),$$

and so the limit is zero.

But $$\int_{(-\frac{1}{2}\pi-\mu)} g(tz)\,e^{-\frac{1}{2}t}\,dt$$

is an analytic function of z when $|z|>\sin\mu$ and

$$\mu<\arg z<\pi+\mu;$$

and so the analytic continuation of $\mathfrak{F}(z)$ over the region in which $|z|>\sin\mu$ and $\mu<\arg z<\pi+\mu$ is given by the equation

$$\mathfrak{F}(z)=\int_{(-\frac{1}{2}\pi-\mu)} g(tz)\,e^{-\frac{1}{2}t}\,dt \quad \ldots\ldots\ldots\ldots(6).$$

Similarly the analytic continuation of $\mathfrak{F}(z)$ over the region in which $|z|>\sin\mu$ and $-\mu>\arg z>-(\pi+\mu)$ is given by the equation

$$\mathfrak{F}(z)=\int_{(\frac{1}{2}\pi+\mu)} g(tz)\,e^{-\frac{1}{2}t}\,dt\ldots\ldots\ldots\ldots\ldots(7).$$

Now let us consider the region in which $\pi\leq\arg z<\frac{3}{2}\pi$. The portion of this region in which $\arg z<\pi+\mu$ and $|z|>\sin\mu$ (where μ is assigned all positive acute values) is the portion which is outside the circle $|z+i|=1$.

Hence when $\pi\leq\arg z<\frac{3}{2}\pi$, $\mathfrak{F}(z)$ has no singularity outside the circle $|z+i|=1$; similarly, when $-\frac{3}{2}\pi<\arg z\leq-\pi$, $\mathfrak{F}(z)$ has no singularity outside the circle $|z-i|=1$.

5. We can now prove that, when $2<|z|<\Delta$, $\mathfrak{F}(z)$ is a uniform function of z; it is supposed that Δ is an arbitrarily large positive number.

Let z_0 be any number such that $2<|z_0|<\Delta$ and $\arg z_0=-\pi$.

Then $$\mathfrak{F}(z_0)=\int_{(\frac{1}{2}\pi+\mu)} g(tz_0)\,e^{-\frac{1}{2}t}\,dt;$$

and the contour may be swung round into the position $\arg t=\frac{1}{2}\pi$, so that

$$\mathfrak{F}(z_0) = \int_{(\frac{1}{2}\pi)} g(tz_0)\, e^{-\frac{1}{2}t}\, dt.$$

Similarly $$\mathfrak{F}(z_0 e^{2\pi i}) = \int_{(-\frac{1}{2}\pi)} g(tz_0 e^{2\pi i})\, e^{-\frac{1}{2}t}\, dt,$$

and writing $te^{2\pi i} = u$, we get

$$\mathfrak{F}(z_0 e^{2\pi i}) = \int_{(\frac{3}{2}\pi)} g(uz_0)\, e^{-\frac{1}{2}u}\, du,$$

so that

$$\mathfrak{F}(z_0 e^{2\pi i}) - \mathfrak{F}(z_0) = \int_{(\frac{3}{2}\pi)} g(tz_0)\, e^{-\frac{1}{2}t}\, dt - \int_{(\frac{1}{2}\pi)} g(tz_0)\, e^{-\frac{1}{2}t}\, dt.$$

The expression on the left is expressible as

$$\lim_{N\to\infty} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} g(tz_0)\, e^{-\frac{1}{2}t}\, \frac{dt}{d\vartheta}\, d\theta,$$

where $t = Ne^{i\theta}$; for since z_0 is negative the integrand is analytic throughout the portion of the t-plane on the left of the imaginary axis, and the deformation of the contour into an infinite semicircle is permissible by Cauchy's theorem.

Now, when $t = Ne^{i\theta}$ and $\frac{1}{2}\pi \leq \theta \leq \frac{3}{2}\pi$ while z_0 is negative, we have

$$|g(tz_0)| < Ae^2 \exp\{-N|z_0|\} < Ae^{2-N},$$

while $$|e^{-\frac{1}{2}t}| < e^{\frac{1}{2}N}.$$

Hence $$|\mathfrak{F}(z_0 e^{2\pi i}) - \mathfrak{F}(z_0)| \leq \lim_{N\to\infty} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} Ae^{2-\frac{1}{2}N}\, N d\theta$$
$$= 0.$$

Therefore $\mathfrak{F}(ze^{2\pi i}) - \mathfrak{F}(z)$ is a function of z which is analytic when $-\frac{3}{2}\pi < \arg z < -\frac{1}{2}\pi$, provided that z lies outside both the circles $|z \pm i| = 1$, and it vanishes all along the line $\arg z = -\pi$ (when $|z| > 1$).

Since the property of being analytic involves the property of being monogenic, it follows that all the derivatives of the function $\mathfrak{F}(ze^{2\pi i}) - \mathfrak{F}(z)$ vanish along the line in question, and so the function is *everywhere* zero, by the general theory of analytic continuation.

Since $\mathfrak{F}(z)$ is analytic and one-valued when $-\frac{3}{2}\pi < \arg z < \frac{3}{2}\pi$ and* $|z| > 2$, it follows from this result that $\mathfrak{F}(z)$ is analytic and one-valued when $\frac{1}{2}\pi < \arg z < \frac{7}{2}\pi$, and so on.

* We take this inequality to ensure that z may be outside the circles $|z \pm i| = 1$.

Consequently $\mathfrak{F}(z)$ is a uniform function of z outside the circle $|z|=2$, and hence it is expansible into a Laurent series when $|z|>2$.

6. We shall now shew that $\mathfrak{F}(z)$ can have no singularities except at the origin and at infinity.

We have seen (§ 4) that $\mathfrak{F}(z)$ has no singularity in the region in which $-\pi<\arg z<\pi$ and $\Delta>|z|>\delta$; and also (§ 5) that $\mathfrak{F}(z)$ is a uniform function which, by § 4, certainly has no singularities in the region on the left of the imaginary axis outside the circles $|z\pm i|=1$.

But all points of the plane, except those in the immediate neighbourhood of the origin and of infinity, belong to one (or both) of these regions and the required result is proved.

7. It can now be shewn that $z\mathfrak{F}(z)$ is a bounded function of z; for if $0\leq\arg z\leq\pi$, we have

$$\mathfrak{F}(z)=\int_{(-\frac{1}{2}\pi)} g(tz)\,e^{-\frac{1}{2}t}\,dt,$$

so that

$$|\mathfrak{F}(z)|<\int_0^\infty Ae^{2-u|z|}\,|e^{\frac{1}{2}iu}|\,du$$

$$=Ae^2\int_0^\infty e^{-u|z|}\,du,$$

which gives $|z\mathfrak{F}(z)|<Ae^2$ when $0\leq\arg z\leq\pi$; we obtain the same inequality when $-\pi\leq\arg z\leq 0$ by taking the formula

$$\mathfrak{F}(z)=\int_{(\frac{1}{2}\pi)} g(tz)\,e^{-\frac{1}{2}t}\,dt.$$

Since a function cannot be bounded* in the neighbourhood of an isolated singularity (whether essential or not), it follows that $z\mathfrak{F}(z)$ is a function which has no singularities at the origin or at infinity. Hence by Liouville's theorem it is a constant.

To determine the constant we observe that we must have

$$z\mathfrak{F}(z)=\lim_{z\to+0}\{z\mathfrak{F}(z)\}.$$

But

$$\mathfrak{F}(+0)=\lim_{z\to+0}\int_0^\infty g(tz)\,e^{-\frac{1}{2}t}\,dt,$$

* Whittaker and Watson, *Modern Analysis*, p. 102.

so
$$|\mathfrak{F}(+0)| \leq \lim_{z\to+0}\left|\int_0^\infty g(tz)\,e^{-\frac{1}{2}t}\,dt\right|$$
$$< Ae^2\int_0^\infty e^{-\frac{1}{2}t}\,dt$$
$$= 2Ae^2;$$

and therefore
$$\lim_{z\to+0}\{z\mathfrak{F}(z)\} = 0.$$

Hence for all values of z we have $z\mathfrak{F}(z) = 0$; and therefore
$$\mathfrak{F}(z) \equiv 0.$$

8. The proof that $g(z)$ is zero is now immediate; for when z is positive we have, on differentiating equation (3),
$$\mathfrak{F}^{(n)}(z) = \int_0^\infty t^n g^{(n)}(tz)\,e^{-\frac{1}{2}t}\,dt \quad (n = 0,\ 1,\ 2,\ \ldots).$$

Since, when $z \geq 0$,
$$|t^n g^{(n)}(tz)\,e^{-\frac{1}{2}t}| \leq n!\,Ae^3\,t^n e^{-\frac{1}{2}t},$$

and
$$\int_0^\infty t^n e^{-\frac{1}{2}t}\,dt$$

is convergent, it follows* that
$$\mathfrak{F}^{(n)}(+0) = \int_0^\infty t^n \lim_{z\to+0} g^{(n)}(tz)\,e^{-\frac{1}{2}t}\,dt,$$

so that
$$0 = g^{(n)}(+0)\int_0^\infty t^n e^{-\frac{1}{2}t}\,dt,$$

i.e. since $g^{(n)}(0) = g^{(n)}(+0)$ because $g(z)$ is monogenic, we have
$$0 = 2^{n+1}\,n!\,g^{(n)}(0) \quad (n = 0,\ 1,\ 2,\ \ldots).$$

It follows at once from Maclaurin's expansion that $g(z) = 0$; and, as shewn in § 2, this necessitates the result that $f(\zeta)$ is zero in the circumstances postulated, namely, that $f(\zeta)$ should be analytic when $R(\zeta) \geq 0$, and should be less, in absolute value, than $A\exp(-|\zeta|)$.

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* Bromwich, *Infinite Series*, pp. 434, 436.

GREEN'S DYADICS IN THE THEORY OF ELASTICITY.

By *C. E. Weatherburn*, *M.A.* (Cambridge), *D.Sc.* (Sydney).

Introduction.

The close analogy which is known to exist between the problems of Dirichlet and Neumann and the fundamental problems of elasticity,* and which in a recent paper† I have further emphasised by the introduction of certain dyadics constructed from Somigliana's integrals of the equations of equilibrium for an elastic isotropic body, suggests a further enquiry as to the existence of other dyadics which, in the theory of elasticity, will play a part similar to that taken by the ordinary Green's functions in the potential problems. In the present paper I show how to construct such dyadics, which I shall call the Green's dyadics for the problems of elasticity corresponding respectively to zero surface displacement and zero surface traction. The former is an ordinary Green's dyadic, and the latter a generalised. Having proved their fundamental properties I show how they may be used for the solution of various problems related to the differential equation of elastic equilibrium under no bodily forces

$$(1) \qquad L(\mathbf{u}) \equiv \nabla^2 \mathbf{u} + k \operatorname{grad} \operatorname{div} \mathbf{u} = 0.$$

In particular I discuss the problems of small vibrations of an isotropic body corresponding to the boundary conditions of zero displacement and zero surface traction; and finally prove bilinear series for the Green's dyadics, corresponding exactly to the ordinary bilinear series for the symmetric kernel of the scalar integral equation. The paper will afford further applications of the theory of vector integral equations, which I have developed in a paper just appearing in the *Transactions of the Cambridge Philosophical Society*; and the important rôle of the conjugo-symmetric kernel there‡ indicated is here filled by the Green's dyadics to be considered.

Though the greater part of the present investigation is independent of my earlier paper on the problems of elasticity,

* Cf. *e.g.* Lauricella, *Atti Lincei* (5), t. 15_1 (1906), pp. 426–433 and 610–619; t. 15_2, pp. 75–83; and *Il Nuovo Cimento* (5), t. 13 (1907), four papers.

† "On two fundamental problems on the theory of elasticity," *Philosophical Magazine*, vol. xxxii., pp. 15–38 (1916).

‡ "Vector integral equations and Gibbs' dyadics," §§ 19–24; *Camb. Phil. Trans.*, vol. xxii., pp. 133–155 (1916).

it will be convenient for the purpose of identifying the functions found in the first section below to mention the following results previously obtained. If $\mathbf{a}$ denote a constant (say unit) vector, then

$$\mathbf{s}_0(pq) = \frac{\mathbf{a}}{r} - \frac{k}{2(1+k)} \operatorname{grad} \operatorname{div} (\mathbf{a}r)$$

is a particular integral of the equation (1) of Somigliana's type, r being measured from a fixed point p (the pole) to the current point q. To this displacement corresponds the surface traction $\mathbf{F}_0(tp)$ at the boundary point t, given by the ordinary formula. With the notation $\mathbf{F}_1(tp)$, $\mathbf{F}_2(tp)$, $\mathbf{F}_3(tp)$ and $\mathbf{s}_1(pq)$, $\mathbf{s}_2(pq)$, $\mathbf{s}_3(pq)$ for the particular values of $\mathbf{F}_0(tp)$ and $\mathbf{s}_0(pq)$ respectively when $\mathbf{a}$ is put equal to the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in succession, the dyadics*

$$\Psi(tp) = (1/2\pi)\,[\mathbf{F}_1(tp)\,\mathbf{i} + \mathbf{F}_2(tp)\,\mathbf{j} + \mathbf{F}_3(tp)\,\mathbf{k}],$$

$$\Phi(pq) = (1/2\pi)\,[\mathbf{i}\,\mathbf{s}_1(pq) + \mathbf{j}\,\mathbf{s}_2(pq) + \mathbf{k}\,\mathbf{s}_3(pq)],$$

form the basis of displacement functions analogous to the vector potentials of double and simple strata respectively. Regarded as kernels of vector integral equations of Fredholm's type with arbitrary parameter λ, the dyadic $\Psi(ts)$ and its conjugate $\chi(ts)$ possess resolvent dyadics $\mathbf{H}(ts)$ and $\mathbf{H}'(ts)$ respectively, connected with them by alternative equations of the form †

$$\mathbf{H}(ts) - \Psi(ts) = \lambda \int \mathbf{H}(t\vartheta) \cdot \Psi(\vartheta s)\, d\vartheta = \lambda \int \Psi(t\vartheta) \cdot \mathbf{H}(\vartheta s)\, d\vartheta,‡$$

and a similar pair for $\mathbf{H}'(ts)$ and $\chi(ts)$. Relations of the same form were shown to be true when the resolvents $\mathbf{H}(ts)$ and $\mathbf{H}'(ts)$ are extended, the second boundary point, s, being replaced by a point p not on the boundary; and lastly two other dyadics $\Gamma(pq)$ and $\Gamma'(pq)$ were defined, connected with the preceding by the equations §

$$(A)\quad \begin{cases} \Gamma(pq) - \Phi(pq) = \lambda \int \Phi(ps) \cdot \mathbf{H}(sq)\, ds = \lambda \int \Gamma(ps) \cdot \Psi(sq)\, ds \\ \Gamma'(pq) - \Phi(pq) = \lambda \int \Phi(ps) \cdot \mathbf{H}'(sq)\, ds = \lambda \int \Gamma'(ps) \cdot \chi(sq)\, ds. \end{cases}$$

In terms of these dyadics the solutions of the first and second boundary problems, which require the determination of displacements $\mathbf{W}(p)$ and $\mathbf{V}(p)$, corresponding respectively to the boundary conditions ‖

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* *Phil. Mag., loc. cit.*, §§ 4–7.

† *Loc. cit.*, §§ 8–9.

‡ The differentials $d\delta$, dt, ds denote elements of the boundary; while dp, dq will be used to indicate elements of volume.

§ *Ibid.*, § 9 (33).

‖ *Ibid.*, § 8 (25).

$$(B)\quad \begin{cases} \frac{1}{2}[\mathbf{W}(t^+) - \mathbf{W}(t^-)] - \frac{1}{2}\lambda[\mathbf{W}(t^+) + \mathbf{W}(t^-)] = \mathbf{f}(t), \\ \frac{1}{2}[\mathbf{TV}(t^-) - \mathbf{TV}(t^+)] - \frac{1}{2}\lambda[\mathbf{TV}(t^-) + \mathbf{TV}(t^+)] = -\mathbf{f}(t), \end{cases}$$

in which $\mathbf{TV}(t^+)$ and $\mathbf{TV}(t^-)$ represent the surface tractions for the inner and outer regions respectively when the displacement is $\mathbf{V}(p)$, are given by the expressions*

$$(C)\quad \begin{cases} \mathbf{W}(p) = \int \mathbf{f}(t) \cdot \mathbf{H}(tp)\, dt, \\ \mathbf{V}(p) = \int \Gamma'(pt) \cdot \mathbf{f}(t)\, dt, \end{cases}$$

except for certain singular values of the parameter λ.

1. Determination and Properties of the Green's Dyadics.

§ 1. *Green's vectors and Green's dyadics.* Considering an isotropic body occupying the inner region S, suppose that we require the displacement at any point p when the surface displacement is $\mathbf{s}_1(qt)$, the pole q being a point of the region S occupied by the body. The required displacement is expressible as a double elastic stratum potential $\mathbf{W}_1(p)$ given by

$$\mathbf{W}_1(p) = \int \mathbf{s}_1(qt) \cdot \mathbf{H}_{-1}(tp)\, dt,$$

a result derivable† from (B) and (C) by putting $\lambda = -1$. Hence the vector function

$$\begin{aligned} \mathbf{G}_1(qp) &= \mathbf{s}_1(qp) - \mathbf{W}_1(p) \\ &= \mathbf{s}_1(qp) - \int \mathbf{s}_1(qt) \cdot \mathbf{H}_{-1}(tp)\, dt \end{aligned}$$

is a solution of (1), regular throughout the region S except at the pole q, where it becomes infinite like i/r, while over the boundary Σ it vanishes identically. Of a similar nature are the functions

$$\mathbf{G}_2(qp) = \mathbf{s}_2(qp) - \int \mathbf{s}_2(qt) \cdot \mathbf{H}_{-1}(tp)\, dt,$$
$$\mathbf{G}_3(qp) = \mathbf{s}_3(qp) - \int \mathbf{s}_3(qt) \cdot \mathbf{H}_{-1}(tp)\, dt.$$

The three vectors $\mathbf{G}_1(qp)$, $\mathbf{G}_2(qp)$, $\mathbf{G}_3(qp)$ may then be called a *set of Green's vectors* for the equation (1) vanishing over the boundary. They represent theoretically the displacements produced in the body when the surface is fixed and a unit force is concentrated at the point q in the directions of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ respectively. Forming with these the dyadic

$$\begin{aligned} \Gamma(qp) &= (1/2\pi)[\mathbf{i}\,\mathbf{G}_1(qp) + \mathbf{j}\,\mathbf{G}_2(qp) + \mathbf{k}\,\mathbf{G}_3(qp)] \\ &= \Phi(qp) - \int \Phi(qt) \cdot \mathbf{H}_{-1}(tp)\, dt, \end{aligned}$$

* *Ibid.*, § 9 (31). † Cf. also *Phil. Mag.*, *loc. cit.*, § 13.

we have an expression which is identical with the dyadic $\Gamma_{-1}(qp)$ given by the equations (A) for the parameter value $\lambda = -1$. As dyadics are subject to the ordinary rules of differentiation and integration, we have in $\Gamma(qp)$ one which, regarded as a function of p, vanishes over the boundary and is a regular solution of (1), except at the point q where its consequents become infinite to the same order as $1/2\pi r$. It will therefore be called the *Green's dyadic* of the equation (1) for the inner region satisfying the boundary condition

$$\Gamma(qt) = 0 \tag{2}$$

of zero surface displacement.

Similarly we have in $\Gamma_{-1}'(pq)$ the Green's dyadic for the outer region S' corresponding to zero surface traction. For if q is a point of that region, the displacement with surface traction $\mathbf{F}_1(tq)$ is given by

$$\mathbf{V}_1(p) = \int \Gamma_{-1}'(pt) \cdot \mathbf{F}_1(tq)\, dt$$

in virtue of (C).* Hence the displacement

$$\mathbf{G}_1(pq) = \mathbf{s}_1(pq) - \int \Gamma_{-1}'(pt) \cdot \mathbf{F}_1(tq)\, dt$$

has zero surface traction, and is of the nature of a Green's vector. Forming with this and the other two vectors of the set the dyadic

$$\begin{aligned}\Gamma'(pq) &= (1/2\pi)[\mathbf{i}\,\mathbf{G}_1(pq) + \mathbf{j}\,\mathbf{G}_2(pq) + \mathbf{k}\,\mathbf{G}_3(pq)] \\ &= \Phi(pq) - \int \Gamma_{-1}'(pt) \cdot \chi(tq)\, dt \\ &= \Gamma_{-1}'(pq),\end{aligned}$$

we have the Green's dyadic for the outer region corresponding to zero surface traction.

§ 2. *The generalized Green's dyadic.* When, however, we try to construct in the above manner a Green's dyadic for the inner region and the condition

$$\mathrm{T}\,\Gamma(qt) = 0 \tag{2'}$$

of zero surface traction, the difficulty of the singular parameter value $\lambda = +1$ confronts us. It is impossible to find a regular displacement satisfying (1) and with surface traction equal to $\mathbf{F}_1(tq)$, for the necessary conditions†

$$\int \mathbf{F}_1(tq)\, dt = 0, \quad \int \rho(t) \times \mathbf{F}_1(tq)\, dt = 0$$

* *Ibid.*, § 16. † *Ibid.*, § 15.

are not satisfied, $\rho(t)$ being the position vector of the point t relative to the c.m. of the body. All that can be done is to find a displacement whose surface traction differs from $\mathbf{F}_1(tq)$ by a function of the form $(\mathbf{a}+\omega\times\rho)$, where $\mathbf{a}$ and ω are constant vectors; and thus construct a Green's dyadic corresponding to surface traction of the form $(\mathbf{a}+\omega\times\rho)$. This is hardly what is required. We rather try to construct a generalised Green's dyadic analogous to the generalised Green's function defined by Hilbert* for the self-adjoint partial differential equation of the second order.

The equation (1) admits regular solutions of the form $\mathbf{a}$ and $\omega\times\rho$ with zero surface traction. In fact it is to this contingency that the non-existence of the ordinary Green's vectors and dyadic is due. These solutions may be normalised so that the *volume* integral of the square of each is equal to unity. In this form they may be written $\mathbf{a}/\sqrt{J}$ and $\omega\times\rho/\sqrt{I}$, where $\mathbf{a}$ and ω are constant *unit* vectors, J the volume of the body, and I the moment of inertia of the volume about the axis through the c.m. parallel to ω. If the principal axes of inertia at the c.m. be taken as the directions of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ respectively, A, B, C being the principal moments of inertia, the six independent normalised solutions of (1) corresponding to zero surface traction are

$$\frac{\mathbf{i}}{\sqrt{J}},\ \frac{\mathbf{j}}{\sqrt{J}},\ \frac{\mathbf{k}}{\sqrt{J}},\ \frac{\mathbf{i}\times\rho}{\sqrt{A}},\ \frac{\mathbf{j}\times\rho}{\sqrt{B}},\ \frac{\mathbf{k}\times\rho}{\sqrt{C}},$$

and these are orthogonal to each other since the products of inertia vanish. Form with these the dyadic

$$\Omega(qp)=\frac{\mathbf{i}\mathbf{i}}{J}+\frac{\mathbf{j}\mathbf{j}}{J}+\frac{\mathbf{k}\mathbf{k}}{J}+\frac{\mathbf{i}\times\rho(q)\,\mathbf{i}\times\rho(p)}{A}+\ldots+$$

$$=\frac{\mathbf{I}}{J}+\frac{\mathbf{i}\times\rho(q)\,\mathbf{i}\times\rho(p)}{A}+\frac{\mathbf{j}\times\rho(q)\,\mathbf{j}\times\rho(p)}{B}+\ldots.$$

Now there exists a solution $\mathbf{G}_1(qp)$ of the differential equation

$$L_p[\mathbf{G}_1(qp)]=\mathbf{i}\cdot\Omega(qp), \tag{3}$$

which becomes infinite at the point $p=q$ in the same manner as an ordinary Green's vector, while it satisfies the boundary condition (2′) of zero surface traction, and is further orthogonal to all the above six orthogonal solutions of (1). This function $\mathbf{G}_1(qp)$ represents theoretically the displacement of the

* "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen," *Zweite Mitt., Gött. Nach.* (1904), s. 219, 238.

point p of the body when a unit force, concentrated at q, acts in the direction of $\mathbf{i}$, while throughout the volume of the body acts a force $-\mathbf{i}\cdot\Omega(qp)$ per unit mass. The displacement due to this system of forces clearly satisfies the differential equation (3) at all points except q; and that the body is in equilibrium under the forces is easily verified* by resolving parallel to $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, and taking moments about the point q. Further, the displacement $\mathbf{G}_1(qp)$ may be chosen so as to be orthogonal to all the six exceptional solutions; for these six conditions are equivalent to the two

$$(4) \qquad \int \mathbf{G}_1(qp)\,dp = 0, \quad \int \mathbf{G}_1(qp)\times\rho(p)\,dp = 0,$$

requiring no translation of the body as a whole, and no rotation about the c.m. These conditions can always be satisfied by the addition of a vector of the form $\mathbf{a}+\omega\times\rho$, representing a displacement of the body as a whole.

Two other vectors $\mathbf{G}_2(qp)$ and $\mathbf{G}_3(qp)$ exist bearing the same relation to the axes of $\mathbf{j}$ and $\mathbf{k}$ respectively that $\mathbf{G}_1(qp)$ bears to $\mathbf{i}$. These three vectors we shall speak of as a set of generalised Green's vectors for the equation (1) and the condition of zero surface traction; while the dyadic

$$\Gamma(qp) = (1/2\pi)\,[\mathbf{i}\,\mathbf{G}_1(qp) + \mathbf{j}\,\mathbf{G}_2(qp) + \mathbf{k}\,\mathbf{G}_3(qp)]$$

formed with them is the corresponding generalised Green's dyadic for the region S. It is clear that $\Gamma(qp)$ satisfies the differential equation

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* To prove that the body is in equilibrium under a unit force $\mathbf{i}$ at q and a bodily force $-\mathbf{i}\cdot\Omega(qp)$ per unit mass throughout. First the resultant of the bodily forces is $-\mathbf{i}$. For

$$\int -\mathbf{i}\cdot\Omega(qp)\,dp = -\int \mathbf{i}\cdot\left[\frac{\mathbf{1}}{J} + \frac{\mathbf{i}\times\rho(q)\,\mathbf{i}\times\rho(p)}{A} + \ldots + \ldots\right] dp$$
$$= -\mathbf{i},$$

since the centroid of the body is the origin of the position vectors $\rho(p)$, making $\int\rho(p)\,dp = 0$. It only remains then to prove that the resultant moment of all the forces is zero about (say) the point q. But the sum of the moments about q

$$= \int\left[\rho(p)-\rho(q)\right]\times\left[\frac{\mathbf{i}}{J} + \frac{\mathbf{j}\times\rho(p)\,\rho(q)\cdot\mathbf{k}}{B} - \frac{\mathbf{k}\times\rho(p)\,\rho(q)\cdot\mathbf{j}}{C}\right] dp$$
$$= \mathbf{i}\times\rho(q) + (1/B)\,\rho(q)\cdot\mathbf{k}\int\rho(p)$$
$$\times[\mathbf{j}\times\rho(p)]\,dp - (1/C)\,\rho(q)\cdot\mathbf{j}\int\rho(p)\times[\mathbf{k}\times\rho(p)]\,dp$$
$$= \mathbf{i}\times\rho(q) + (1/B)\,\rho(q)\cdot\mathbf{k}\,[\tfrac{1}{2}(A+B+C)\,\mathbf{j} - \tfrac{1}{2}(A-B+C)\,\mathbf{j}] - \ldots$$
$$= \mathbf{i}\times\rho(q) + \rho(q)\cdot\mathbf{k}\,\mathbf{j} - \rho(q)\cdot\mathbf{j}\,\mathbf{k} = 0.$$

The argument has depended upon the fact that the products of inertia vanish, since the directions of the principal axes through the centroid are those of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

$$L_p[\Gamma(qp) = (1/2\pi)\,\Omega(qp), \tag{5}$$

and, corresponding to (4), the integral relations

$$\int \Gamma(qp)\,dp = 0, \quad \int \Gamma(qp) \times \rho(p)\,dp = 0. \tag{6}$$

§ 3. *Conjugo-symmetry.* Let $\mathbf{u}$ and $\mathbf{v}$ be two functions of position, regular within the region considered, and, when regarded as displacements, having surface tractions $\mathbf{Tu}$ and $\mathbf{Tv}$ respectively. Then Betti's theorem expresses that*

$$\int [L(\mathbf{u}) \cdot \mathbf{v} - \mathbf{u} \cdot L(\mathbf{v})]\,dp = \int [\mathbf{Tu} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{Tv}]\,dt. \tag{7}$$

If in this we put for $\mathbf{u}$ and $\mathbf{v}$ the Green's vectors $\mathbf{G}_1(qp)$ and $\mathbf{G}_1(q'p)$, with poles at q and q' respectively, and possessing either both zero surface displacement or both zero surface traction, we must exclude the poles by (say) small spherical surfaces Z and Z' in the usual manner. The volume integral then vanishes identically; for the ordinary Green's vectors satisfy (1), and though the generalised vectors satisfy (3), yet in virtue of the relations (4) the volume integral in (7) still vanishes. Moreover, the surface integral over Σ disappears because of the boundary conditions satisfied by these vectors. The equation then reduces to†

$$\int_{Z+Z'} [\mathbf{G}_1(q's) \cdot \mathbf{F}_1(sq) - \mathbf{G}_1(qs) \cdot \mathbf{F}_1(sq')]\,ds = 0.$$

Similarly if in (7) we substitute in turn the pairs of vectors $\mathbf{G}_1(qp)$, $\mathbf{G}_2(q'p)$ and $\mathbf{G}_1(qp)$, $\mathbf{G}_3(q'p)$ satisfying the same boundary condition as before, we obtain the equations

$$\int_{Z+Z'} [\mathbf{G}_2(q's) \cdot \mathbf{F}_1(sq) - \mathbf{G}_1(qs) \cdot \mathbf{F}_2(sq')]\,ds = 0,$$

$$\int_{Z+Z'} [\mathbf{G}_3(q's) \cdot \mathbf{F}_1(sq) - \mathbf{G}_1(qs) \cdot \mathbf{F}_3(sq')]\,ds = 0.$$

Multiplying these equations by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively and adding, we find, after division by 2π,

$$\int_{Z+Z'} [\Gamma(q's) \cdot \mathbf{F}_1(sq) - \mathbf{G}_1(qs) \cdot \Psi(sq')]\,ds = 0,$$

where $\Gamma(qp)$ is the Green's dyadic for the boundary condition considered. Two equations similar to this may be written down

* Cf. Betti, *Il Nuovo Cimento* (1872); *Annali di Mat.*, t. 6 (1875); also Lévy, *C. R.*, t. 107 (1888), pp. 413 and 453.

† By thus writing the surface traction of $\mathbf{G}_1(qp)$ at Z as $\mathbf{F}_1(sq)$ we are neglecting the traction due to the regular function $\mathbf{W}_1(p)$ of § 1. But when the surface Z becomes vanishingly small the part of the integral due to this also vanishes.

by altering the suffix. Then taking the indeterminate products of the three equations by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ respectively as consequents, and adding, we have a result which may be written

$$\int_{Z+Z'} [\Gamma(q's) \cdot \Psi(sq) - \Psi_c(sq') \cdot \Gamma_c(qs)]\, ds = 0.$$

Now let the radii of the spheres Z and Z' decrease indefinitely; then in the limit we find, exactly as in § 4 of my earlier paper,

$$\Gamma(q'q) = \Gamma_c(qq'). \tag{8}$$

Thus the Green's dyadics are conjugo-symmetric* with properties analogous to those of the symmetric kernel in the ordinary integral equation.

§ 4. *A fundamental formula.* If, however, in Betti's theorem (7) we put for $\mathbf{u}$ any regular solution of the non-homogeneous equation

$$L[\mathbf{u}(p)] = -2\,\mathbf{f}(p), \tag{10}$$

where $\mathbf{f}(p)$ is continuous, and for $\mathbf{v}$ a Green's vector $\mathbf{G}_1(qp)$ with pole at q we must isolate this point by a small surface Z in the usual manner. In the case of the ordinary Green's vectors satisfying (1) we thus obtain

$$-2\int \mathbf{G}_1(qp) \cdot \mathbf{f}(p)\, dp = \int_{Z+\Sigma'} [\mathbf{G}_1(qs) \cdot \mathbf{T}\mathbf{u}(s) - \mathbf{u}(s) \cdot \mathbf{T}\mathbf{G}_1(qs)]\, ds,$$

while two similar equations may be written down in $\mathbf{G}_2(qs)$ and $\mathbf{G}_3(qs)$. Multiplying these in order by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and adding, and then making the surface Z decrease indefinitely, we find in the limit by the same argument as above that

$$\mathbf{u}(q) = \int \Gamma(qp) \cdot \mathbf{f}(p)\, dp - \tfrac{1}{2}\int [\mathbf{u}(t) \cdot \mathbf{T}\Gamma(qt) - \Gamma(qt) \cdot \mathbf{T}\mathbf{u}(t)]\, dt.\dagger \tag{11}$$

Now the Green's dyadic is that of zero surface displacement; and therefore if $\mathbf{u}$ is that solution of (10) which vanishes over the boundary, the surface integral in the last formula disappears, leaving us with the result

$$\mathbf{u}(q) = \int \Gamma(qp) \cdot \mathbf{f}(p)\, dp, \tag{12}$$

which gives a source-representation for the function $\mathbf{u}(q)$. Hence

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* Cf. "Vector integral equations, &c.", §§ 19—24.

† By $\mathbf{T}\Gamma$ is obviously meant the result obtained by performing on Γ the same operation as we perform on $\mathbf{u}$ to find $\mathbf{T}\mathbf{u}$.

THEOREM I. Any displacement function $\mathbf{u}(q)$ which is continuous along with its derivatives of the first two orders, and vanishes over the boundary of the region S, is capable of a source-representation as in the last formula.

The solution of (10) which vanishes over the boundary is thus given uniquely by (12); and we have proved the formula

$$L\int\Gamma(qp)\cdot\mathbf{f}(p)\,dp=-2\,\mathbf{f}(q). \tag{13}$$

When the boundary condition of zero surface traction is under consideration, we use the generalised Green's vectors satisfying equations of the form (3); and in place of (12) we now find for that solution of (10) which gives zero surface traction

$$\mathbf{u}(q)=\int\Gamma(qp)\cdot\mathbf{f}(p)\,dp+(1/4\pi)\int\Omega(qp)\cdot\mathbf{u}(p)\,dp. \tag{14}$$

If then $\mathbf{u}(p)$ also satisfies the same relations (4) as the generalised Green's vectors, this formula becomes as before

$$\mathbf{u}(q)=\int\Gamma(qp)\cdot\mathbf{f}(p)\,dp. \tag{14'}$$

As pointed out in § 2, $\mathbf{u}(p)$ can always be chosen so as to satisfy these conditions by the addition of a suitable displacement of the form $(\mathbf{a}+\omega\times\rho)$. Hence

THEOREM II. Any displacement $\mathbf{u}(p)$ which is continuous along with its derivatives of the first two orders, and produces zero surface traction while it further satisfies the relations (4), is capable of source-representation in terms of the generalised Green's dyadic as in (14').

It follows from the preceding argument that the Green's dyadics are *closed kernels*. For if there exists a continuous function $\mathbf{f}(p)$ such that

$$\int\Gamma(qp)\cdot\mathbf{f}(p)\,dp=0,$$

we deduce from (13) that

$$L\int\Gamma(qp)\cdot\mathbf{f}(p)\,dp=-2\,\mathbf{f}(q)=0.$$

Hence $\mathbf{f}(q)$ vanishes identically, and the Green's dyadic is a closed conjugo-symmetric kernel. It follows therefore that its characteristic numbers are *real* and *infinitely many*.* Further, denoting the normal characteristic functions of $\Gamma(qp)$ by $\phi_i(p)$, we have the following theorems:

* *Ibid.*, § 23, Theorem 3.

THEOREM III. If $\mathbf{F}(p)$ is a continuous function, and the series

$$\sum_{i=1}^{\infty} \phi_i(p) \int \mathbf{F}(q) \cdot \phi_i(q)\, dq$$

converges uniformly, it represents the function $\mathbf{F}(p)$.*

THEOREM IV. Any displacement function $\mathbf{u}(p)$ which is continuous along with its derivatives of the first two orders, which further satisfies the same boundary condition as the Green's dyadic $\Gamma(pq)$, and in the case of the generalised dyadic the further relations (4), can be expanded by the Fourier rule in an absolutely and uniformly convergent series† in terms of the characteristic functions $\phi_i(p)$.

§ 5. *An equation related to* $L(\mathbf{u}) = 0$. Suppose that we require a regular solution of the equation

$$L(\mathbf{u}) + 2\lambda \mathbf{u} = 0 \tag{16}$$

satisfying the same boundary condition as the Green's dyadic $\Gamma(qp)$, and in the case of the generalised dyadic the relations (4) also; then from the preceding results it is given by the integral equation

$$\mathbf{u}(p) = \lambda \int \Gamma(pq) \cdot \mathbf{u}(q)\, dq, \tag{17}$$

which admits a non-zero solution only when λ is equal to one of the characteristic numbers λ_i of the kernel $\Gamma(pq)$. The corresponding solutions are the characteristic functions $\phi_i(p)$ of this kernel. Hence the differential equation (16) admits the required solution only for these particular values of λ, such solutions being the characteristic functions of the integral equation (17). We may thus speak of these solutions and parameter values as the characteristic functions and parameter values of the differential equation (16). Their number has been proved infinite.

It can be shown, as in the case of the scalar self-adjoint partial differential equation,‡ that the resolvent of $\Gamma(pq)$ is the Green's dyadic for the differential equation (16) satisfying the same boundary condition. This dyadic therefore exists for all values of λ except the characteristic values of $\Gamma(pq)$; but at each of these the Green's dyadic of (16) becomes infinite, having a simple pole, as is the case with the resolvent of any conjugo-symmetric dyadic kernel.§

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* Cf. Hilbert, "Grundzüge," *Erste Mitt.*, Kap. IV., Satz 6.
† "Vector integral equations, &c.," § 23, Theorem 2.
‡ Hilbert, *loc. cit.*, s. 224, 239.
§ "Vector integral equations," § 20, Theorem 4.

It is important for our purpose to show that the characteristic numbers λ_i are all positive. Take the form*

$$(18) \qquad \int \{\Pi[\mathbf{u}(p)] + \mathbf{u} \cdot L(\mathbf{u})\}\, dp = \int \mathbf{u} \cdot \mathbf{T}\mathbf{u}\, dt$$

of Betti's theorem, in which $\Pi(\mathbf{u})$ denotes the potential due to the displacement $\mathbf{u}$. If in this we put for $\mathbf{u}$ a solution of (16) satisfying either of the boundary conditions, the second member vanishes, while the first becomes

$$\int \Pi[\mathbf{u}(p)]\, dp - \lambda \int \mathbf{u}^2 dp.$$

Since the first integral is a positive function representing twice the potential energy of the deformed body, it follows that λ must be positive in order that the whole expression may vanish. Hence the characteristic numbers of the Green's dyadics are all positive.

II. Equilibrium of an elastic isotropic body.

§ 6. *No bodily forces.* The solution of the *first* boundary problem considered in my earlier paper, requiring the determination of a regular integral $\mathbf{u}(p)$ of (1) when its boundary value $\mathbf{u}(t)$ is known is given as a particular case of the formula (11). For taking $\Gamma(qp)$ as the Green's dyadic for zero surface displacement, and putting $\mathbf{f}(p)$ equal to zero, we obtain the required solution in the form

$$(19) \qquad \begin{cases} \mathbf{u}(p) = -\frac{1}{2} \int \mathbf{u}(t) \cdot \mathbf{T}\Gamma(t^+ p)\, dt \\ \qquad\quad = \int \mathbf{u}(t) \cdot \mathbf{H}_{-1}(tp)\, dt, \dagger \end{cases}$$

which is identical with the result previously found.‡

In the *second* boundary problem we are given the value of the surface traction. Taking then in (11) $\Gamma(qp)$ as the generalised Green's dyadic for zero surface traction, and putting $\mathbf{f}(p) = 0$, we obtain the formula

$$(19') \qquad \mathbf{u}(p) = \tfrac{1}{2} \int \Gamma(pt^+) \cdot \mathbf{T}\mathbf{u}(t)\, dt,$$

where the displacement $\mathbf{u}(p)$ has been chosen to satisfy the conditions (4) as already explained. This is identical in form with the solution found previously,§ but the dyadic $\Gamma(pt)$ is not the same as there employed.

A *third* boundary problem for the equation (1) requiring a displacement satisfying the relation

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* Cf. Betti, *loc. cit.*

† Cf. *Phil. Mag.*, *loc. cit.*, formula (50).

‡ *Ibid.*, § 13.

§ *Ibid.*, § 15.

$$\mathbf{T}\,\mathbf{u}(t) = \lambda\,\mathbf{u}(t) - \mathbf{U}(t),$$

where $\mathbf{U}(t)$ is a given function of the position t, may be treated along lines exactly parallel to those followed by Plemelj* in the corresponding problem for Laplace's equation. The surface traction for a given displacement takes the place of normal derivative, Betti's theorem is used instead of Green's, while in place of the additive constant C there introduced we have a vector of the form $(\mathbf{a} + \omega \times \rho)$, the constant vectors $\mathbf{a}$ and ω being determined so that the solution satisfies the relations (4).

§ 7. *General problem.* We may also write down the solution of the general problem requiring the displacement $\mathbf{u}(p)$ at any point of an isotropic body under the bodily force $\mathbf{P}(p)$ per unit mass, the boundary condition being either one of those already treated, viz.

$$(2) \qquad \begin{cases} \text{(a)} & \mathbf{u}(t) = 0, \\ \text{(b)} & \mathbf{T}\,\mathbf{u}(t) = 0, \end{cases}$$

or one of the non-homogeneous conditions

$$(20) \qquad \begin{cases} \text{(a)} & \mathbf{u}(t) = \mathbf{U}(t), \\ \text{(b)} & \mathbf{T}\,\mathbf{u}(t) = \mathbf{V}(t). \end{cases}$$

The differential equation fo be satisfied is

$$(21) \qquad L[\mathbf{u}(p)] = -\mathbf{P}(p).$$

From the argument of § 4 it follows that if the displacement satisfies one of the homogeneous boundary relations (2) it is given by

$$\mathbf{u}(p) = \tfrac{1}{2}\int \Gamma(pq) \cdot \mathbf{P}(q)\, dq,$$

with the usual precaution in the case of zero surface traction.

When, however, the boundary displacement has the non-zero value given by (20a), the formula (11) becomes, in terms of the Green's dyadic for zero surface displacement,

$$\mathbf{u}(p) = \tfrac{1}{2}\int \Gamma(pq) \cdot \mathbf{P}(q)\, dq - \tfrac{1}{2}\int \mathbf{U}(t) \cdot \mathbf{T}\,\Gamma(tp)\, dt.$$

The first integral is the solution of (21) vanishing over the boundary; the second is the integral of (1) with surface value $\mathbf{U}(t)$.

Similarly for the condition (20b) when the surface traction is $\mathbf{V}(t)$, we use the generalised Green's dyadic and (11) becomes

* *Monatshefte für Math. und Physik*, Bd. 18 (1907), s. 180—211.

$$\mathbf{u}(p) = \tfrac{1}{2}\int \Gamma(pq)\cdot\mathbf{P}(q)\,dq + \tfrac{1}{2}\int \Gamma(pt)\cdot\mathbf{V}(t)\,dt.$$

For the equilibrium of the body the bodily force and the surface traction must satisfy the relations

$$(22)\qquad \begin{cases} \int \mathbf{P}(p)\,dp + \int \mathbf{V}(t)\,dt = 0, \\ \int \rho(p)\times\mathbf{P}(p)\,dp + \int \rho(t)\times\mathbf{V}(t)\,dt = 0. \end{cases}$$

III. Small vibrations of an elastic isotropic body.

§ 8. *General problem of free vibrations.* We shall now consider the small free vibrations of an elastic body occupying the region S, the boundary condition being either that of zero surface displacement or that of zero surface traction. The differential equation satisfied by the displacement $\mathbf{u}(p)$ is, with suitable choice of units,*

$$(23)\qquad L(\mathbf{u}) = \frac{\partial^2}{\partial t^2}\mathbf{u}.$$

For harmonic vibrations we make the ordinary substitution

$$\mathbf{u}(p, t) = \mathbf{v}(p)\begin{cases}\cos kt \\ \sin kt\end{cases},$$

in which $\mathbf{v}(p)$ is independent of the time, and thus obtain the differential equation

$$L[\mathbf{v}(p)] = -k^2\mathbf{v}(p),$$

while the boundary condition for $\mathbf{v}(p)$ is the same as for $\mathbf{u}(p)$. If $\Gamma(qp)$ be the Green's dyadic for this condition, these are equivalent to the homogeneous integral equation

$$(24)\qquad \mathbf{v}(p) = \lambda\int \Gamma(pq)\cdot\mathbf{v}(q)\,dq, \quad (\lambda = \tfrac{1}{2}k^2),$$

the function $\mathbf{v}(p)$ in the case of zero surface traction satisfying the conditions (4). This integral equation admits a non-zero solution only when λ is equal to one of the characteristic numbers λ_i of the Green's dyadic; and as these have been proved all positive, the values of k found from them are all real. If $\mathbf{v}_i(p)$ are the characteristic functions of the equation (24), the solution of the problem thus takes the form

$$(25)\qquad \mathbf{u}(p, t) = \sum_{i=1}^{\infty}\mathbf{v}_i(p)\left[A_i\cos\sqrt{(2\lambda_i)}\,t + B_i\sin\sqrt{(2\lambda_i)}\,t\right].$$

* Cf., *e.g.*, Love, *Mathematical Theory of Elasticity*, vol. i., § 29, first edition.

It is clear that the constants A_i and B_i are substantially the coefficients in the expansions of the functions $\alpha(p)$ and $\beta(p)$ representing the initial values of the displacement $\mathbf{u}$ and the velocity $\dot{\mathbf{u}}$ in terms of the functions $\mathbf{v}_i(p)$. By Theorem IV. (§ 4) these initial functions may be expanded in terms of the normal functions $\mathbf{v}_i(p)$ by the Fourier rule

$$(26) \qquad \begin{cases} \alpha(p) = \sum_{i=1}^{\infty} \mathbf{v}_i(p) \int \alpha(q) \cdot \mathbf{v}_i(q)\, dq \\ \beta(p) = \sum_{i} \mathbf{v}_i(p) \int \beta(q) \cdot \mathbf{v}_i(q)\, dq, \end{cases}$$

and thus the coefficients A_i and B_i are determined. In the case of zero surface traction, unless the functions $\alpha(p)$ and $\beta(p)$ satisfy the conditions (4), we must determine vectors $\mathbf{a}_1$, $\mathbf{a}_2$, ω_1 and ω_2 in such a way that the functions

$$\alpha'(p) = \alpha(p) + \mathbf{a}_1 + \omega_1 \times \rho(p),$$
$$\beta'(p) = \beta(p) + \mathbf{a}_2 + \omega_2 \times \rho(p)$$

do satisfy the conditions (4). Then these functions may be expanded as in (26), giving the required series for $\alpha(p)$ and $\beta(p)$.

§ 9. *Forced vibrations.* In the problem of forced vibrations the natural period of the body is set aside or overpowered by that of the forces controlling the motion. Suppose these forces to be of the harmonic type, their value at any point p being $\mathbf{F}(p)\cos kt$ per unit mass. Then the differential equation of motion is

$$(27) \qquad L(\mathbf{u}) = \frac{\partial^2}{\partial t^2}\mathbf{u} - \mathbf{F}(p)\cos kt,$$

the boundary condition being as before. Since the vibration of each particle is compelled to keep in step with the force, the required displacement $\mathbf{u}$ will be of the form

$$\mathbf{u}(p, t) = \mathbf{v}(p)\cos kt,$$

$\mathbf{v}(p)$ being independent of t. Making this substitution we find for $\mathbf{v}(p)$ the differential equation

$$L[\mathbf{v}(p)] = -k^2\mathbf{v}(p) - \mathbf{F}(p)$$

with the original boundary condition. If $\Gamma(pq)$ is the appropriate Green's dyadic, these are equivalent to the non-homogeneous integral equation

$$\mathbf{v}(p) = \lambda \int \Gamma(pq) \cdot \mathbf{v}(q)\, dq + \mathbf{f}(p),$$

in which $\mathbf{f}(p) = \frac{1}{2}\int \Gamma(pq) \cdot \mathbf{F}(q)\, dq,$

and $\lambda = \frac{1}{2}k^2.$

The kernel of this equation is conjugo-symmetric, and the solution may be expressed by the extension of Schmidt's formula*

$$\mathbf{v}(p) = \mathbf{f}(p) + \lambda \sum_{n=1}^{\infty} C_n \mathbf{v}_n(p)/(\lambda_n - \lambda),$$

where λ_n, $\mathbf{v}_n(p)$ are the characteristic numbers and normal function of the kernel $\Gamma(pq)$, and

$$C_n = \int \mathbf{f}(q) \cdot \mathbf{v}_n(q)\, dq.$$

But from the form of $\mathbf{f}(p)$ it may be expanded† in terms of the normal functions in the series

$$\mathbf{f}(p) = \sum_{n=1}^{\infty} C_n \mathbf{v}_n(p).$$

The value of $\mathbf{v}(p)$ then becomes

$$\mathbf{v}(p) = \sum_n C_n \lambda_n \mathbf{v}_n(p)/(\lambda_n - \lambda),$$

giving for the forced vibrations of the system

$$(28) \qquad \mathbf{u}(p, t) = \cos kt \sum_n C_n \lambda_n \mathbf{v}_n(p)/(\lambda_n - \lambda).$$

The *most general vibration* of the body under the conditions of the problem consists of two parts, and may be expressed

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2,$$

where $\mathbf{u}_1$ represents the forced vibrations just found, and $\mathbf{u}_2$ the free vibrations superimposed by the former. The latter satisfies the homogeneous equation (23). Hence for the most general solution of (27) we have

(29)

$$\mathbf{u}(p, t) = \sum_{n=1}^{\infty} \mathbf{v}_n(p) \left[A_n \cos(k_n t) + B_n \sin(k_n t) + \frac{C_n \lambda_n}{\lambda_n - \lambda} \cos kt \right],$$

where, as before, $k_n^2 = 2\lambda_n.$

Let $\alpha(p)$ and $\beta(p)$ be the initial values of $\mathbf{u}$ and $\dot{\mathbf{u}}$ respectively, with expansions represented by (26). Comparing these with the values of the same quantities obtained from (29) we find

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* "Vector integral equations, &c.," § 23, Theorem 4.
† *Ibid.*, § 23, Theorem 2.

$$\begin{cases} A_n = -\,C_n\lambda_n/(\lambda_n-\lambda)+\int\alpha(p)\cdot\mathbf{v}_n(p)\,dp \\ B_n = (1/k_n)\int\beta(p)\cdot\mathbf{v}_n(p)\,dp. \end{cases}$$

(30)

$$\begin{cases} \mathbf{u}(p,\,t)=\sum_{n=1}^{\infty}\mathbf{v}_n(p)\left[\dfrac{C_n\lambda_n}{\lambda_n-\lambda}(\cos kt-\cos k_n t)\right. \\ \qquad \left.+\cos k_n t\int\alpha(p)\cdot\mathbf{v}_n(p)\,dp+\dfrac{1}{k_n}\sin k_n t\int\beta(p)\cdot\mathbf{v}_n(p)\,dp\right]. \end{cases}$$

§ 10. *Damped vibrations.* If the vibrating body is under the influence of dissipative forces such as are represented by the addition of a term* of the form $w\cdot d\mathbf{u}/dt$ to the second member of (24), the vibrations being otherwise free, we have to find a solution of the differential equation

$$L(\mathbf{u})=\frac{\partial^2}{\partial t^2}\mathbf{u}+w\frac{\partial}{\partial t}\mathbf{u} \tag{31}$$

satisfying the same boundary condition (2) as in the previous cases. Assuming w constant for the isotropic body, make the substitution

$$\mathbf{u}(p,\,t)=\mathbf{v}(p)\,e^{(in-w/2)t}.$$

The equation (32) then becomes

$$L[\mathbf{v}(p)]=-2\lambda\mathbf{v}(p), \tag{32}$$

where we have put

$$\lambda=(4n^2+w^2)/8;$$

while the boundary condition is unaltered. If $\Gamma(pq)$ is the appropriate Green's dyadic, the function $\mathbf{v}(p)$ is given by

$$\mathbf{v}(p)=\lambda\int\Gamma(pq)\cdot\mathbf{v}(q)\,dq.$$

This integral equation admits solutions $\mathbf{v}_i(p)$ only for the characteristic parameter values λ_i, which are all positive, and in terms of which the corresponding values of n are

$$n_i=\tfrac{1}{2}\sqrt{(8\lambda_i-w^2)}.$$

At the most then a finite number of the n's are imaginary. If the dissipative forces are sufficiently small, all the n's are real. The solution of the problem is given by

* A resistance proportional to the velocity. It is doubtful if this has any physical significance for a solid body. However, I propose the problem for what it is worth.

$$(33)\quad \mathbf{u}(p, t) = e^{-wt/2} \sum_{i=1}^{\infty} \mathbf{v}_i(p)\,[A_i \cos n_i t + B_i \sin n_i t],$$

the constants A_i and B_i being determined from the expansions of the functions $\alpha(p)$ and $\beta(p)$ representing the initial condition of the body.

§ 11. *Bilinear dyadic series.* We have already given a physical interpretation to the Green's vectors. The relation of the Green's dyadics to the bilinear series may be seen from the following mechanical considerations,* which at the same time show the physical significance of the characteristic functions.

Let q_n ($n = 1, 2, \ldots, \infty$) be a set of parameters, in the sense of the Lagrangean mechanics, expressing the possible positions of the rigid body, each chosen so as to vanish in the position of equilibrium. On the assumption that the vibrations are simple harmonic we may put

$$(34)\qquad \mathbf{u}(p, t) = \sum_n q_n \mathbf{v}_n(p),$$

the parameters q_n being functions of the time satisfying differential equations of the form

$$\ddot{q}_n + \lambda_n q_n = 0.$$

This necessitates the kinetic energy T being free from product terms $\dot{q}_n \dot{q}_m (m \neq n)$, so that

$$T = \tfrac{1}{2} \sum_n \dot{q}_n^2, \quad V = \tfrac{1}{2} \sum_n \lambda_n q_n^2;$$

V being the potential energy. We might have started with these forms; for "when we have two homogeneous quadratic functions of any number of variables, one of which is essentially positive for all values of the variables, it is known that by a real linear transformation of the variables we may clear both expressions of the terms containing the products of the variables, and also make the coefficients of the squares in the positive function each equal to unity or some given positive constants."† From (35), however, we find for the kinetic energy

$$T = \tfrac{1}{2} \int [\sum_n \dot{q}_n \mathbf{v}_n(p)]^2 dp,$$

* Cf., *e.g.*, Kneser, "Integralgleichungen," §§ 9, 16. Also Lord Rayleigh, "Theory of Sound," vol. i, chapter v.

† Routh, "Elementary rigid dynamics," § 459.

dp being the element of mass at p, or the element of volume since the body is of uniform density. Identifying these values of the kinetic energy, we find

$$(35) \qquad \int \mathbf{v}_n(p) \cdot \mathbf{v}_m(p)\, dp \begin{cases} = 1, & \text{if } m = n, \\ = 0, & \text{if } m \neq n. \end{cases}$$

These are the orthogonal relations satisfied by the normalised characteristic functions.

Suppose now that the body is vibrating under a force-system R, the force on the element dp being $\mathbf{P}(p)\, dp$. Then the virtual work of these forces due to a small displacement $\delta\mathbf{u}$ is

$$\int \delta\mathbf{u} \cdot \mathbf{P}(p)\, dp = \sum_n \delta q_n \int \mathbf{v}_n(p) \cdot \mathbf{P}(p)\, dp$$

$$= \sum_n Q_n \delta q_n,$$

where we have put

$$(36) \qquad Q_n = \int \mathbf{v}_n(p) \cdot \mathbf{P}(p)\, dp.$$

The equation of vibration under the force-system R is then

$$\ddot{q}_n + \lambda_n q_n = Q_n.$$

If, however, we imagine the body in equilibrium under these forces the term $\ddot{q}_n$ is zero, and $q_n = Qn/\lambda_n$. The equilibrium displacement is then given by

$$(37) \qquad \mathbf{u}(p) = \sum_n Q_n \mathbf{v}_n(p)/\lambda_n.$$

If the boundary condition is that of *zero surface displacement*, the force system R may, consistently with the equilibrium of the body, be concentrated into a unit force $\mathbf{a}$ at the point q, plus the surface traction necessary to keep the boundary fixed. The latter, however, does not appear in the equation of virtual work. The function Q_n now takes the form $\mathbf{a} \cdot \mathbf{v}_n(q)$, and the displacement $\mathbf{G}(qp)$ under these forces is

$$(38) \qquad \begin{cases} \mathbf{G}(qp) = \sum_n \mathbf{a} \cdot \mathbf{v}_n(q)\mathbf{v}_n(p)/\lambda_n = \mathbf{a} \cdot \left[\sum_n \dfrac{\mathbf{v}_n(q)\mathbf{v}_n(p)}{\lambda_n}\right] \\ \qquad\qquad = \mathbf{a} \cdot \Theta(qp), \end{cases}$$

where the dyadic $\Theta(qp)$ represents the bilinear series. Suppose we give the unit vector $\mathbf{a}$ the values $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in succession; the corresponding displacements are the Green's vector $\mathbf{G}_1(qp)$, $\mathbf{G}_2(qp)$, $\mathbf{G}_3(qp)$ respectively. Forming with these the Green's dyadic $\Gamma(qp)$ we have

$$(39)\qquad \begin{cases} \Gamma(qp) = (1/2\pi)\,[\mathbf{ii}+\mathbf{jj}+\mathbf{kk}]\cdot\Theta(qp) \\ \qquad\quad = (1/2\pi)\,\Theta(qp), \end{cases}$$

that is, the Green's dyadic is represented to a constant factor by the bilinear series.* That this series is convergent follows by Mercer's theorem† from the fact that the characteristic numbers are all positive. Hence the *bilinear formula* (40) is true. On multiplying scalarly both sides of this formula by $\lambda_n \mathbf{v}_n(q)$, and integrating with respect to q over the region S, we find, in virtue of the relations (36), the integral equation

$$(40)\qquad \mathbf{v}_n(p) = \lambda_n \int \Theta(pq)\cdot\mathbf{v}_n(q)\,dq$$

satisfied by the characteristic functions $\mathbf{v}_n(p)$.

§ 12. In the case of *zero surface traction*, however, $\lambda_o = 0$ is a characteristic value to which correspond the six normalised characteristic functions $\mathbf{v}_{oi}(p)$, $(i = 1, 2, \ldots, 6)$, equal respectively to

$$\frac{\mathbf{i}}{\sqrt{J}},\ \frac{\mathbf{j}}{\sqrt{J}},\ \frac{\mathbf{k}}{\sqrt{J}},\ \frac{\mathbf{i}\times\rho}{\sqrt{A}},\ \frac{\mathbf{j}\times\rho}{\sqrt{B}},\ \frac{\mathbf{k}\times\rho}{\sqrt{C}}.$$

The kinetic energy may be written

$$T = \tfrac{1}{2}\sum_{i=1}^{6} \dot{q}_{oi}^{\,2} + \tfrac{1}{2}\Sigma\,\dot{q}_n^{\,2},$$

while from the expression for the potential energy the first six terms $\frac{1}{2}\sum_{i=1}^{6} \lambda_o q_{oi}^{\,2}$ disappear. The displacement $\mathbf{u}(p)$ becomes

$$\mathbf{u}(p, t) = \sum_{i=1}^{6} q_{oi}\mathbf{v}_{oi}(p) + \Sigma\, q_n \mathbf{v}_n(p).$$

Corresponding to the parameter q_{oi} Lagrange's equation of motion under the force system R is

$$\ddot{q}_{oi} = Q_{oi};$$

and therefore when there is equilibrium under these forces

$$O = Q_{oi} = \int \mathbf{P}(p)\cdot\mathbf{v}_{oi}(p)\,dp,$$

a condition which will not in general be satisfied when the system R is concentrated as a unit force at the point q.

* $\Gamma(qp)$ and $\Theta(qp)$ have the same normal functions $\mathbf{v}_n(p)$, but the characteristic numbers of $\Gamma(qp)$ are 2π times those of $\Theta(qp)$.

† *Phil. Trans. Roy. Soc.*, vol. ccix. (A) (1909), pp. 415—446.

Suppose, however, that we introduce at each element of mass dp the additional force

$$-\mathbf{a}\cdot\Omega(qp)\,dp=-\mathbf{a}\cdot\Big[\sum_{i=1}^{6}\mathrm{V}_{oi}(q)\mathrm{V}_{oi}(p)\Big]dp,$$

in which $\mathbf{a}$ is a *unit* vector. Then the above condition becomes

$$O=-\int\mathbf{a}\cdot\Omega(qp)\cdot\mathrm{V}_{oi}(p)\,dp+\int\mathbf{P}(p)\cdot\mathrm{V}_{oi}(p)\,dp$$
$$=-\mathbf{a}\cdot\mathrm{V}_{oi}(q)+\int\mathbf{P}(p)\cdot\mathrm{V}_{oi}(p)\,dp,$$

which is satisfied when the force system R is concentrated as a unit force $\mathbf{a}$ at the point q. Then, as in the preceding case, Q_n takes the form $\mathbf{a}\cdot\mathrm{V}_n(q)$, and the displacement $G(qp)$ under the combined forces is

$$\mathbf{G}(qp)=\sum_{i=1}^{6}q_{oi}\mathrm{V}_{oi}(p)+\mathbf{a}\cdot\Big[\sum_n\frac{\mathrm{V}_n(q)\mathrm{V}_n(p)}{\lambda_n}\Big].$$

We can choose this displacement as explained in § 2, so that all the quantities $q_{oi}(i=1, 2, \ldots, 6)$ disappear. For the particular unit vectors $\mathbf{a}=\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively the above displacement then becomes identical with the generalised Green's vectors. Forming with them the generalised dyadic $\Gamma(qp)$ we have again the bilinear formula

$$(39')\qquad\Gamma(qp)=\sum_n\frac{\mathrm{V}_n(q)\mathrm{V}_n(p)}{2\pi\lambda_n}.$$

The dyadic series is convergent as before, because the values λ_n are all positive, and the bilinear formula is true. Further, from the orthogonal relations (26), it is clear that the generalised Green's vectors as just found satisfy the relations

$$\int\mathbf{G}(qp)\cdot\mathrm{V}_{oi}(p)\,dp=0\quad(i=1, 2, \ldots, 6),$$

while on account of the extra bodily force $-\mathbf{a}\cdot\Omega(qp)$ per unit mass, the differential equations satisfied by them are of the form (3). From (40′) it follows, as above, that the characteristic functions satisfy the integral equation

$$\mathrm{V}_n(p)=2\pi\lambda_n\int\Gamma(pq)\cdot\mathrm{V}_n(q)\,dq.$$

ON A CERTAIN DEFINITE INTEGRAL.

By *C. V. Hanumanta Rao.*

Mr. Hardy has evaluated the integral

$$\int_0^\infty \sin x \sin\left(\frac{b}{x}\right) \frac{dx}{x^s}$$

in the *Messenger of Mathematics* for July 1910. I propose to evaluate the integral

$$I(b) = \int_0^\infty J_m(x)\, J_n\left(\frac{b}{x}\right) \frac{dx}{x^s},$$

following the same line of argument. The integral converges provided $-n - \frac{1}{2} < s < m + \frac{5}{2}$. We may (under somewhat narrower conditions) differentiate the integral with respect to b.* We obtain

$$I_1 = \int_0^\infty J_m(x)\, J_n'\left(\frac{b}{x}\right) \frac{dx}{x^{s+1}},$$

the suffix denoting differentiation with respect to b. Integrating this equation by parts, we get

$$I_1 = \frac{1}{b} \int_0^\infty J_n\left(\frac{b}{x}\right) \left\{ \frac{J_m'(x)}{x^{s-1}} - (s-1) \frac{J_m(x)}{x^s} \right\} dx,$$

the extra term vanishing whenever $-n + \frac{1}{2} < s < m + \frac{3}{2}$. Thus

$$bI_1 + (s-1) I = \int_0^\infty J_m'(x)\, J_n\left(\frac{b}{x}\right) \frac{dx}{x^{s-1}}.$$

Differentiating again, and integrating by parts, we obtain

$$bI_2 + sI_1 = \frac{1}{b} \int_0^\infty J_n\left(\frac{b}{x}\right) \left\{ \frac{J_m''(x)}{x^{s-2}} - (s-2) \frac{J_m'(x)}{x^{s-1}} \right\} dx.$$

Substituting for $J_m''(x)$ from the equation satisfied by $J_m(x)$ and re-arranging, it follows that

$$b^2 I_2 + bI_1 (2s-1) + \{(s-1)^2 - m^2\} I = -\int_0^\infty J_m(x)\, J_n\left(\frac{b}{x}\right) \frac{dx}{x^{s-2}} \quad \ldots\ldots\ldots (A).$$

* The differentiation may be justified by arguments similar to those used by Mr. Hardy. The same remark applies to a number of similar processes to be used later.

Again, writing $G(b)=\int_0^\infty J_m(x)\,J_n\left(\frac{b}{x}\right)\frac{dx}{x^{s-2}}$,

we obtain $G_1(b)=\int_0^\infty J_m(x)\,J_n'\left(\frac{b}{x}\right)\frac{dx}{x^{s-1}}$,

and $G_2(b)=\int_0^\infty J_m(x)\,J_n''\left(\frac{b}{x}\right)\frac{dx}{x^s}$.

Substituting for $J_n''(b/x)$ in terms of $J_n(b/x)$ and $J_n'(b/x)$, and making use of the previous equalities, we obtain

$$b^2G_2+bG_1-n^2G=-b^2I\ldots\ldots\ldots\ldots\ldots\ldots(B).$$

Eliminating G and its differential coefficients from (A) and (B), we get an equation for I of the fourth degree, viz.

$$b^4I_4+b^3I_3(2s+4)+b^2I_2(s^2+4s+2-m^2-n^2)$$
$$+bI_1(s^3-2n^2s-m^2+n^2)-n^2I\{(s-1)^2-m^2\}=b^2I.$$

Writing $s=1-\nu$, four independent solutions of this equation are found to be A, B, C and D, where

$$A=b^{m+\nu}\sum_{k=0}^{\infty}\frac{b^{2k}}{2^{4k}\Gamma(1+k)\Gamma(1+k+m)\Gamma\{1+k+\frac{1}{2}(\nu+m+n)\}\Gamma\{1+k+\frac{1}{2}(\nu+m-n)\}},$$

$$B=b^{n}\sum_{k=0}^{\infty}\frac{b^{2k}}{2^{4k}\Gamma(1+k)\Gamma(1+k+n)\Gamma\{1+k-\frac{1}{2}(\nu+m-n)\}\Gamma\{1+k-\frac{1}{2}(\nu-m-n)\}},$$

and C and D are obtained by changing the sign of m in A and that of n in B. Hence

$$I(b)=c_1A+c_2B+c_3C+c_4D,$$

where the c's are constants. It remains to determine these constants.

When b is small,

$$A\sim b^{m+\nu}/[\Gamma(1+m)\Gamma\{1+\tfrac{1}{2}(\nu+m+n)\}\Gamma\{1+\tfrac{1}{2}(\nu+m-n)\}]$$

and

$$B\sim b^n/\ [\Gamma(1+n)\Gamma\{1-\tfrac{1}{2}(\nu+m-n)\}\Gamma\{1-\tfrac{1}{2}(\nu-m-n)\}].$$

Also $$I\sim\frac{b^n}{2^n n!}\int_0^\infty J_m(x)\frac{dx}{x^{1-\nu+n}}$$
$$=\frac{b^n}{2^n n!}\,\frac{2^{q-m-1}\Gamma(\frac{1}{2}q)}{\Gamma(1+m-\frac{1}{2}q)},$$

where $q=m-n+\nu$, provided $0<q<m+\frac{3}{2}$. It follows that

$$I \sim \frac{2^{\nu-1-\frac{1}{2}n}\pi b^n}{\sin\{\frac{1}{2}(m-n+\nu)\pi\}\Gamma(1+n)\,\Gamma\{1-\frac{1}{2}(m-n+\nu)\}\,\Gamma\{1+\frac{1}{2}(m+n-\nu)\}}.$$

Hence $c_3 = c_4 = 0$, provided $n - m < \nu < n + m$ and $o < n$.

Thus $$I = c_1 A + c_2 B,$$

where $$c_2 = \frac{\pi}{2^{\frac{1}{2}m+s}\sin\{\frac{1}{2}(m-n+\nu)\pi\}}.$$

It remains to determine c_1. Consider the behaviour of the series as $b \to \infty$. It is known that,* when b is large,

$$A \sim b^{\frac{1}{2}\nu-\frac{1}{4}} 2^{2m+\nu-1}\pi^{-\frac{3}{2}} e^{2b^{\frac{1}{2}}},$$

and $$B \sim b^{\frac{1}{2}\nu-\frac{3}{4}} 2^{2n-\nu-1}\pi^{-\frac{3}{2}} e^{2b^{\frac{1}{2}}}.$$

Again $$|I| < cb^{-\frac{1}{4}} \int_0^\infty |J_m(x)|\,\frac{dx}{x^{s-\frac{1}{4}}},$$

where c is a constant, for all positive values of b, provided $0 < s < m + \frac{3}{2}$. We thus find that

$$b^{\frac{1}{2}\nu-\frac{1}{4}} e^{2b^{\frac{1}{2}}}(c_1 2^{\nu+2m} + c_2 2^{-\nu+2n})$$

is less than a constant, from which it results that

$$c_1 2^{\nu+2m} + c_2 2^{-\nu+2n} = 0.$$

Finally we obtain

$$I = \frac{\pi}{2\sin\{\frac{1}{2}(m-n+\nu)\pi\}}\left\{\frac{B}{2^{2n-\nu}} - \frac{A}{2^{2m+\nu}}\right\}.$$

In the notation of Dr. Barnes this may be written as

$$I = \frac{\pi}{2\sin\{\frac{1}{2}(m-n+\nu)\pi\}}\left\{\frac{-b^{m+\nu}P}{d_1} + \frac{b^n Q}{d_2}\right\},$$

where

$$d_1 = 2^{2m+\nu}\Gamma(1+m)\,\Gamma\{1+\tfrac{1}{2}(\nu+m+n)\}\,\Gamma\{1+\tfrac{1}{2}(\nu+m-n)\},$$

$$d_2 = 2^{n-\nu}\Gamma(1+n)\,\Gamma\{1-\tfrac{1}{2}(\nu+m-n)\}\,\Gamma\{1-\tfrac{1}{2}(\nu-m-n)\},$$

$$P = {}_1F_4\left\{1;\ 1,\ 1+m,\ 1+\frac{\nu+m+n}{2},\ 1+\frac{\nu+m-n}{2};\ \frac{b^2}{16}\right\},$$

and

$$Q = {}_1F_4\left\{1;\ 1,\ 1+n,\ 1-\frac{\nu+m-n}{2},\ 1-\frac{\nu-m-n}{2};\ \frac{b^2}{16}\right\}.$$

The formula for I has been obtained on the supposition that

* See Barnes, *Proc. London Math. Soc.*, ser. 2, vol. v. (1907), p. 62.

$$n - m < \nu < n + m,$$

$$-\tfrac{1}{2} - m < \nu < n + \tfrac{1}{2},$$

$$0 < n.$$

Changing x into b/x, we see that the first of these conditions may as well be

$$-n - m < \nu < n - m.$$

Hence the formula holds for

$$-n - m < \nu < m + n.$$

But I itself exists for

$$-\tfrac{3}{2} - m < \nu < n + \tfrac{3}{2}.$$

It is not difficult to prove that the formula obtained holds throughout this range except at a number of points where the formulæ become illusory. Taking $m = n = \frac{1}{2}$ and observing that

$$J_{\frac{1}{2}}(x) = \sqrt{\left(\frac{2}{\pi x}\right)} \sin x,$$

it is easy to verify that we get Mr. Hardy's result. In the following further cases the expression obtained for I reduces to the sum of a number of Bessel functions:—

$$\nu = \quad 0, \quad n = m \pm 1;$$

$$m = \quad n, \quad \nu = \quad \pm 1;$$

$$m = -n, \quad \nu = \quad \pm 1.$$

It may be remarked that whenever

$$\int_0^\infty J_m(x)\, J_n\left(\frac{b}{x}\right) \frac{dx}{x^s}$$

can be expressed as the sum of a number of Bessel functions, the same is true of

$$\int_0^\infty J_m(x)\, J_{n+p}\left(\frac{b}{x}\right) \frac{dx}{x^{s-p}},$$

where p is any integer, positive or negative, the range of p being determined by the need for the resulting integrals to converge.

THE CLASS NUMBER FOR DEFINITE BINARY QUADRATICS.

By *L. J. Mordell*, Birkbeck College, London.

One of the most important questions in the Theory of Numbers, to find the number of classes of binary quadratics of given discriminant, was originally solved by Gauss, but the first published solution was given by Dirichlet, and is now to be found in most books on the subject. His results were afterwards considerably simplified by Kronecker. The method employed depends upon finding the limit of a series* of the form $\rho \sum_{n=1}^{\infty} a_n^{1+\rho}$ when $\rho \to 0$. As far as I am aware no other method has been discovered, so that the following solution when the discriminant is negative may prove of interest.

The quadratics are taken in the form $ax^2+hxy+by^2$, where $h^2-4ab=D=-n$ and n is positive. Moreover it is assumed that D is a fundamental discriminant† (the general case is easily reduced to this), so that improper forms do not exist. Also $\left(\frac{p}{q}\right)$ is the symbol of quadratic reciprocity as generalized by Kronecker.

It is well known‡ that the total number of representations of any given number M by all the non-equivalent forms of discriminant D is equal to $\tau \sum_d \left(\frac{D}{d}\right)$, wherein d refers to all the divisors of M and

$$\tau=2 \text{ if } D<-4,$$

$$\tau=4 \text{ if } D=-4,$$

$$\tau=6 \text{ if } D=-3.$$

This result can be written as (if $|r|<1$)

$$\sum_{a,b,c} \sum_{x,y} r^{ax^2+hxy+by^2} = A + \tau \sum_{k=1}^{\infty} \left(\frac{D}{k}\right) \frac{r^k}{1-r^k} \quad \ldots(1),$$

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* Series of the type Σa_n^{ρ} have attracted considerable attention lately, apart from their application in the Theory of Numbers. See for example "The General Theory of Dirichlet's Series" by Hardy and Riesz.

† J. de Seguier, *Formes Quadratiques*, pp. 10, 25, 33. This book is hereafter referred to as Seguier.

‡ Seguier, p. 112.

where x, y take all integer values, positive, negative and zero, and the summation with respect to a, b, c refers to all the non-equivalent binary quadratics of discriminant D. Moreover A is a constant which is obviously equal to the number of classes of forms of discriminant D. I find it by putting $r = e^{2\pi\iota\omega}$, where $I(\omega) > 0$ and then substituting $-1/\omega$ for ω in equation (1), which I write for shortness, as

$$\phi(\omega) = A + \chi(\omega).$$

We know* that

$$\phi\left(-\frac{1}{\omega}\right) = \frac{\omega}{\sqrt{(-n)}}\,\phi\left(\frac{\omega}{n}\right),$$

where $$\sqrt{(-n)} = \sqrt{(D)} = \iota\sqrt{(n)}.$$

To find $\chi(-1/\omega)$ we first evaluate $\chi(\omega)$ in terms of theta functions. Writing $\theta(v)$ or $\theta(v, \omega)$ instead of

$$\theta_{11}(v, \omega) = 2r^{\frac{1}{8}} \sin\pi v \prod_{t=1}^{\infty} (1 - r^t)^2 (1 - 2r^t \cos 2\pi v + r^{2t}),$$

and, differentiating logarithmically, we find

$$\frac{\theta'(v)}{\theta(v)} = \pi \cot\pi v + 4\pi \sum_{t=1}^{\infty} \frac{r^t \sin 2\pi v}{1 - 2r^t \cos 2\pi v + r^{2t}}$$

$$= \pi \cot\pi v + \frac{4\pi r}{1-r} \sin 2\pi v + \frac{4\pi r^2}{1-r^2} \sin 4\pi v + \dots .$$

Put now $$f(\omega) = \sum_{v=1}^{n-1} \left(\frac{D}{v}\right) \theta'\left(\frac{v}{n}\right) \Big/ \theta\left(\frac{v}{n}\right),$$

and we have

$$f(\omega) = \pi \sum_{v=1}^{n-1} \left(\frac{D}{v}\right) \cot\left(\frac{\pi v}{n}\right) + 4\pi \sum_{v=1}^{n-1} \sum_{t=1}^{\infty} \left(\frac{D}{v}\right) \frac{r^t}{1-r^t} \sin\left(\frac{2\pi v t}{n}\right).$$

But

$$\sum_{v=1}^{n-1} \left(\frac{D}{v}\right) \sin\left(\frac{2\pi v t}{n}\right) = \frac{1}{2\iota} \sum_{v=1}^{n-1} \left(\frac{D}{v}\right) \left[e^{2\pi\iota v t/n} - e^{-2\pi\iota v t/n}\right],$$

and it is known† that

$$\sum_{v=1}^{n-1} \left(\frac{D}{v}\right) e^{2vh\pi\iota/n} = \pm \left(\frac{D}{h}\right) \sqrt{(D)},$$

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* See my paper, "Theta Functions in the Theory of the Modular Functions..." in the *Quarterly Journal*, vol. xlvi. (1915), p. 105.

† Seguier, p. 32.

according as $h \gtrless 0$, so that

$$\sum_{v=1}^{n-1}\left(\frac{D}{v}\right)\sin\left(\frac{2\pi vt}{n}\right)=\frac{1}{\iota}\left(\frac{D}{t}\right)\sqrt{(D)}=\left(\frac{D}{t}\right)\sqrt{(n)}.$$

Hence

$$f(\omega)=\pi\sum_{v=1}^{n-1}\left(\frac{D}{v}\right)\cot\left(\frac{\pi v}{n}\right)+4\pi\sqrt{(n)}\sum_{t=1}^{\infty}\left(\frac{D}{t}\right)\frac{r^t}{1-r^t},$$

giving us the value of $\chi(\omega)$.

We now investigate the value of $f(-1/\omega)$ in terms of $f(\omega)$.

From the definition

$$f\left(-\frac{1}{\omega}\right)=\sum_{v=1}^{n-1}\left(\frac{D}{v}\right)\theta'\left(\frac{v}{n},\ -\frac{1}{\omega}\right)\Big/\theta\left(\frac{v}{n},\ -\frac{1}{\omega}\right).$$

But $$\theta\left(\frac{\xi}{\omega},\ -\frac{1}{\omega}\right)=-\iota\sqrt{(-\iota\omega)}\,e^{\pi\iota\xi^2/\omega}\,\theta(\xi,\ \omega),$$

so that

$$\theta'\left(\frac{\xi}{\omega},\ -\frac{1}{\omega}\right)\Big/\theta\left(\frac{\xi}{\omega},\ -\frac{1}{\omega}\right)=2\pi\iota\xi+\omega\theta'(\xi,\ \omega)/\theta(\xi,\ \omega).$$

Hence

$$f\left(-\frac{1}{\omega}\right)=\sum_{v=1}^{n-1}\left(\frac{D}{v}\right)\left[\frac{2\pi\iota v\omega}{n}+\omega\theta'\left(\frac{v\omega}{n}\right)\Big/\theta\left(\frac{v\omega}{n}\right)\right]$$

$$=\sum_{v=1}^{n-1}\frac{2\pi\iota\omega v}{n}\left(\frac{D}{v}\right)+\omega\sum_{v=1}^{n-1}\left(\frac{D}{v}\right)\left[\pi\cot\left(\frac{\pi\omega v}{n}\right)\right.$$

$$\left.+4\pi\sum_{t=1}^{\infty}\frac{r^t}{1-r^t}\sin\left(\frac{2\pi v\omega t}{n}\right)\right].$$

But $2\iota\sin x=e^{\iota x}-e^{-\iota x}$ and $\cot x=\iota(e^{2\iota x}+1)/(e^{2\iota x}-1)$,

so that

$$f\left(-\frac{1}{\omega}\right)=2\pi\iota\sum_{v=1}^{n-1}\frac{\omega v}{n}\left(\frac{D}{v}\right)-\omega\pi\iota\left\{\sum_{v=1}^{n-1}\left(\frac{D}{v}\right)\frac{1+r^{v/n}}{1-r^{v/n}}\right.$$

$$\left.+2\sum_{v=1}^{n-1}\sum_{t=1}^{\infty}\left(\frac{D}{v}\right)\left(\frac{r^{t+tv/n}}{1-r^t}-\frac{r^{t-tv/n}}{1-r^t}\right)\right\}.$$

Now, since

$$\frac{1+r^{v/n}}{1-r^{v/n}}=1+\frac{2r^{v/n}}{1-r^{v/n}},\quad\sum_{v=1}^{n-1}\left(\frac{D}{v}\right)=0,$$

and $$\frac{r^{t+tv/n}}{1-r^t} = \sum_{s=0}^{\infty} r^{t+tv/n+st},$$

we have, by writing $n\omega$ for ω,

$$f\left(-\frac{1}{n\omega}\right) = 2\pi\iota \sum_{v=1}^{n-1} \omega v \left(\frac{D}{v}\right) - n\omega\pi\iota \left[\sum_{v=1}^{n-1} 2 \left(\frac{D}{v}\right) \frac{r^v}{1-r^v} \right.$$

$$\left. + 2 \sum_{v=1}^{n-1} \sum_{t=1}^{\infty} \sum_{s=0}^{\infty} \left(\frac{D}{v}\right) \left(r^{nt+tv+snt} - r^{nt-tv+snt}\right)\right].$$

Summing with respect to t, this becomes

$$f\left(-\frac{1}{n\omega}\right) = 2\pi\iota \sum_{v=1}^{n-1} \omega v \left(\frac{D}{v}\right) - n\omega\pi\iota \left[\sum_{v=1}^{n-1} 2 \left(\frac{D}{v}\right) \frac{r^v}{1-r^v} \right.$$

$$\left. + 2 \sum_{v=1}^{n-1} \sum_{s=0}^{\infty} \left(\frac{D}{v}\right) \left(\frac{r^{n+v+sn}}{1-r^{n+v+sn}} - \frac{r^{n-v+sn}}{1-r^{n-v+sn}}\right)\right].$$

The part multiplied by $-n\omega\pi\iota$ can be written as

$$2 \sum_{v=1}^{n-1} \sum_{s=0}^{\infty} \left(\frac{D}{v}\right) \frac{r^{v+sn}}{1-r^{v+sn}} - 2 \sum_{v=1}^{n-1} \sum_{s=0}^{\infty} \left(\frac{D}{v}\right) \frac{r^{n-v+sn}}{1-r^{n-v+sn}},$$

but* since

$$\left(\frac{D}{v}\right) = \left(\frac{D}{v+sn}\right) = -\left(\frac{D}{n-v+sn}\right), \quad \text{and} \quad \left(\frac{D}{M}\right) = 0$$

if M is not prime to n, this is the same as

$$2 \sum_{M=1}^{\infty} \left(\frac{D}{M}\right) \frac{r^M}{1-r^M} + 2 \sum_{M=1}^{\infty} \left(\frac{D}{M}\right) \frac{r^M}{1-r^M}.$$

Hence

$$f\left(-\frac{1}{n\omega}\right) = 2\pi\iota\omega \sum_{v=1}^{n-1} v \left(\frac{D}{v}\right) - 4n\omega\pi\iota \sum_{M=1}^{\infty} \left(\frac{D}{M}\right) \frac{r^M}{1-r^M}$$

$$= 2\pi\iota\omega \sum_{v=1}^{n-1} v \left(\frac{D}{v}\right) - \frac{4n\omega\pi\iota}{4\pi\sqrt{(n)}} \left\{ f(\omega) - \pi \sum_{v=1}^{n-1} \left(\frac{D}{v}\right) \cot\left(\frac{\pi v}{n}\right) \right\}.$$

Put now $\omega = \iota/\sqrt{(n)}$ so that $-1/n\omega = \omega$, hence

$$2 \sum_{v=1}^{n-1} v \left(\frac{D}{v}\right) + \sqrt{(n)} \sum_{v=1}^{n-1} \left(\frac{D}{v}\right) \cot\left(\frac{\pi v}{n}\right) = 0 \ldots\ldots (2),$$

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* Seguier, p. 27.

and so $$f(-1/n\omega) = -\iota\omega\sqrt{(n)}f(\omega),$$

from which $$f\left(-\frac{1}{\omega}\right) = \frac{\omega}{\sqrt{(-n)}}f\left(\frac{\omega}{n}\right).$$

Writing equation (1) in the form

$$\phi(\omega) = A + \frac{\tau}{4\pi\sqrt{(n)}}\left\{f(\omega) + \frac{2\pi}{\sqrt{(n)}}\sum_{v=1}^{n-1} v\left(\frac{D}{v}\right)\right\},$$

and changing ω into $-1/\omega$, we have evidently

$$A + \frac{\tau}{2n}\sum_{v=1}^{n-1} v\left(\frac{D}{v}\right) = 0,$$

which is the result as given by Kronecker.*

We note also that

$$\sum_{a,b,c}\sum_{x,y} r^{ax^2+hxy+by^2} = \frac{\tau}{4\pi\sqrt{(n)}}\left\{\sum_{v=1}^{n-1}\left(\frac{D}{v}\right)\theta'\left(\frac{v}{n}\right) \Big/ \theta\left(\frac{v}{n}\right)\right\},$$

a result which I find is given by Weber in the *Goettingen Nachrichten* for 1893. He finds it by making use of the formula for the class number and equation (2), which is due to Lebesgue.†

ON THE SOLUTIONS OF $x^2+y^2+z^2+t^2=4m_1m_2$.

By *L. J. Mordell*, Birkbeck College, London.

If m_1 is a given odd number, it is well known that the number of solutions of the equation $x_1^2+y_1^2+z_1^2+t_1^2=4m_1$, where x_1, y_1, z_1, t_1 are all odd and taken with such a sign that $x_1\equiv y_1\equiv z_1\equiv t_1\equiv 1 \pmod 4$ (unless otherwise mentioned, the same restriction applies to all equations of this kind) is equal to the sum of the divisors of m_1. It follows at once that if m_1 and m_2 are two odd numbers prime to each other, the number of solutions of $x^2+y^2+z^2+t^2=4m_1m_2$ is equal to the product of the number of solutions of $x_1^2+y_1^2+z_1^2+t_1^2=4m_1$ by the number of solutions of $x_2^2+y_2^2+z_2^2+t_2^2=4m_2$. But from a pair of solutions x_1, y_1, z_1, t_1 and x_2, y_2, z_2, t_2, it is well known that we can derive a solution x, y, z, t from

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* Seguier, p. 126.

† Liouville (1856), p. 392.

$$\left.\begin{aligned} 2x &= x_1x_2-y_1y_2-z_1z_2-t_1t_2 \\ 2y &= x_1y_2+y_1x_2+z_1t_2-t_1z_2 \\ 2z &= x_1z_2-y_1t_2+z_1x_2+t_1y_2 \\ 2t &= x_1t_2+y_1z_2-z_1y_2+t_1x_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(1).$$

For evidently x, y, z, t are odd integers, and we can take them with such a sign that they are all $\equiv 1 \pmod 4$; also

$$4(x^2+y^2+z^2+t^2)=(x_1^2+y_1^2+z_1^2+t_1^2)(x_2^2+y_2^2+z_2^2+t_2^2).$$

The question arises whether all the values of x, y, z, t can be obtained in this way. This is the case. In one of his papers* Dr. Glaisher says, "I have little or no doubt of the truth of this theorem, but I have not obtained a proof. I have not, however, thought it desirable to devote much time to the subject from the belief that the theorem (being of so elementary a character) must have been already proved, though probably in a different form." Having occasion recently to make use of this theorem, and finding no reference to any proof, I invented the following one.

We can establish a unique correspondence between a solution x_1, y_1, z_1, t_1, and the linear substitution

$$A_1=\begin{vmatrix} \tfrac{1}{2}(x_1+\iota y_1), & \tfrac{1}{2}(z_1+\iota t_1) \\ \tfrac{1}{2}(-z_1+\iota t_1), & \tfrac{1}{2}(x_2-\iota y_2 \end{vmatrix}$$

of determinant m_1, and similarly for A_2, A from x_2, etc., and x, etc. But if we compound A_1 and A_2, we find

$$A_1A_2=\begin{vmatrix} \tfrac{1}{2}(x+\iota y), & \tfrac{1}{2}(z+\iota t) \\ \tfrac{1}{2}(-z+\iota t), & \tfrac{1}{2}(x-\iota y) \end{vmatrix}=A,$$

since

$$2(x+\iota y)=(x_1+\iota y_1)(x_2+\iota y_2)+(z_1+\iota t_1)(-z_2+\iota t_2)$$

$$2(z+\iota t)=(x_1+\iota y_1)(z_2+\iota t_2)+(z_1+\iota t_1)(x_2-\iota y_2).$$

Hence we have to prove, that if

$$A_1A_2=B_1B_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where B_1 is the same as A_1, except that x_1, y_1, z_1, t_1 are replaced by another solution $x_1^1, y_1^1, z_1^1, t_1^1$, of $x_1^2+y_1^2+z_1^2+t_1^2=4m_1$, and similarly for B_2, that $A_1=B_1$ and $A_2=B_2$. The proof is immediate, for $A_1^{-1}B_1=A_2B_2^{-1}$ where A_1^{-1} is the inverse

* "The arithmetical functions $P(m)$, $Q(m)$, $\Omega(m)$," *Quarterly Journal*, vol. xxxvii. (1905), p. 48.

substitution of A_1. Now $A_1^{-1}B_1$ is a linear substitution of determinant unity, whose coefficients are fractions with a common denominator $4m_1$, and $A_2B_2^{-1}$ is a linear substitution of determinant unity, whose coefficients are fractions with a common denominator $4m_2$. But since m_1 is prime to m_2, these coefficients must be integers divided by 4. Hence we can write $B_1=A_1S$, where S is defined by

$$\begin{vmatrix} \frac{1}{4}(a+\iota b), & \frac{1}{4}(c+\iota d) \\ \frac{1}{4}(-c+\iota d), & \frac{1}{4}(a-\iota b) \end{vmatrix},$$

and a, b, c, d are any integers satisfying $a^2+b^2+c^2+d^2=16$. Hence $a=\pm 4$, $b=c=d=0$, or $b=\pm 4$, $a=c=d=0$, etc., or $a=\pm 2$, $b=\pm 2$, $c=\pm 2$, $d=\pm 2$. But since from $B_1=A_1S$

$$4x_1'=x_1a-y_1b-z_1c-t_1d,$$

$$4y_1'=x_1b+y_1a+z_1d-t_1c,$$

etc.,

we must reject all the values of a, b, c, d except $a=4$, $b=c=d=0$, as otherwise x_1', y_1', etc., cannot be congruent to 1 (mod 4). For example, if $a=b=c=d=2$, we find

$$x_1'+y_1'+z_1'+t_1'=2x_1.$$

Hence we have $A_1=B_1$ and $A_2=B_2$.

We still have to shew that equations (1) cannot give sets of values such as x, y, z, t; $-x$, y, z, t; $-x$, $-y$, z, t, etc. Writing $1+(x_1-1)$ for x, etc., and noting that $x_1\equiv 1 \pmod 4$, we find $-x\equiv y\equiv z\equiv t \pmod 4$. Hence we only have to rule out the possibility that equations (1) can give sets of values such as x, y, z, t and $-x$, $-y$, $-z$, $-t$. If the set $-x$, $-y$, $-z$, $-t$ could occur, we should have, instead of (2), $A_1A_2T=B_1B_2$, where T is the substitution

$$\begin{vmatrix} -1, & 0 \\ 0, & -1 \end{vmatrix}.$$

This would give as before $B_1=A_1S$, and we must reject all the values of a, b, c, d except $a=4$, $b=c=d=0$. This set must also be rejected by considering $A_2T=SB_2$, as it makes $x_2'=-x_2$, and we are given that $x_1\equiv x_2\equiv 1 \pmod 4$.

This completes the proof of the original theorem.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

XLVIII.

On some properties of integrals of fractional order.

1. Suppose that $f(x)$ is summable throughout any interval of positive values of x, $x=0$ included, and that $\alpha > 0$. Then $F_\alpha(x)$, the integral of f, from 0 to x, of order α, may be defined, in accordance with the ideas of Liouville and Riemann, by the equation

$$(1) \qquad F_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt.$$

It is easily verified that, when α is an integer k,

$$F_k(x) = \int_0^x dt \int^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{k-2}} f(t_{k-1})\, dt_{k-1},$$

so that (1) agrees with the ordinary definition. Also*

$$F_{\alpha+1}(x) = \int_0^x F_\alpha(t)\, dt.$$

We may therefore, when investigating the properties of integrals of fractional order, confine ourselves to the case in which $0 < \alpha < 1$.

A fundamental theorem asserts that

$$F(x) = F_1(x) = \int_0^x f(t)\, dt$$

is continuous. My object in this note is to investigate certain analogues of this theorem for integrals of fractional order. My ideas are in great measure borrowed from those used by Prof. W. H. Young in certain investigations in the theory of Fourier series, referring particularly to generalisations of 'Parseval's Theorem' and 'the Riesz-Fischer Theorem'. The point of view is different but, except that Prof. Young deals with integrals of the form

$$\int_a^b g(x+t) f(t)\, dt$$

* See Theorem (i) below.

and I with integrals of the form

$$\int_0^x g(x-t)f(t)\,dt,$$

the arguments are much the same.

2. I begin by noting two almost obvious theorems.

(i) *$F_\alpha(x)$ exists for almost all values of x, and is summable.*

This follows from the equations

$$\begin{aligned}\int_0^x F_\alpha(t)\,dt &= \frac{1}{\Gamma(\alpha)}\int_0^x dt\int_0^t (t-u)^{\alpha-1}f(u)\,du\\ &= \frac{1}{\Gamma(\alpha)}\int_0^x f(u)\,du\int_u^x (t-u)^{\alpha-1}\,dt\\ &= \frac{1}{\Gamma(\alpha+1)}\int_0^x (x-u)^\alpha f(u)\,du;\end{aligned}$$

in which, in accordance with the classical theorems of de la Vallée-Poussin and Lebesgue, the existence of any one of the integrals which occur involves the existence and the equality of all.

(ii) *If $F_\alpha(x)$ exists for a particular value of x, then $F_\beta(x)$, where $\beta>\alpha$, also exists for that value of x, and is a continuous function of β.*

This follows at once from the equation

$$F_\beta(x)=\int_0^x (x-t)^{\beta-\alpha}(x-t)^{\alpha-1}f(t)\,dt.$$

3. I pass now to the principal results.

(iii) *Suppose that f^{1+p}, where $p>0$, is summable, and that $0<\alpha<1/(1+p)$. Then F_α^ρ is summable for all values of ρ less than*

$$1\Big/\Big(\frac{1}{1+p}-\alpha\Big).$$

(iv) *If $\alpha>1/(1+p)$, then F_α is continuous.*

It is plain that, in proving these theorems, we may suppose without loss of generality that f is positive.

I base the proof of Theorem (iii) on the following simple but important lemma due to Prof. Young.*

* W. H. Young, 'Sur la généralisation du théorème de Parseval', *Comptes Rendus*, 1 July 1912.

If $u(x) \geq 0$, $v(x) \geq 0$, $p \geq 0$, $q \geq 0$, *and* $pq < 1$, *then*

$$(\int uv\,dx)^{\{(1+p)(1+q)\}/(1-pq)}$$

$$\leq \int u^{1+p} v^{1+q}\,dx\,(\int u^{1+p}\,dx)^{\{q(1+p)\}/(1-pq)}\,(\int v^{1+q}\,dx)^{\{p(1+q)\}/(1-pq)},$$

whatever the interval of integration, provided only that the right-hand side has a meaning.

Suppose first that $p > 0$, $q > 0$, and write

$$u^{1+p} = U, \quad v^{1+q} = V,$$

$$a = \frac{(1+p)(1+q)}{1-pq}, \quad \varpi = p + q + 2pq,$$

so that $a > 1$ and

$$\frac{a}{a-1} = \frac{(1+p)(1+q)}{\varpi}.$$

Then $$uv = (UV)^{1/a}\,U^{q/(1+p)}\,V^{p/(1+p)};$$

and so, by the 'generalised inequality of Schwarz', viz.

$$\int \phi\psi\,dx \leq (\int \phi^k\,dx)^{1/k}\,(\int \psi^{k/(k-1)}\,dx)^{k/(k-1)},$$

we obtain

$$\int uv\,dx \leq (\int UV\,dx)^{1/a}\,(\int U^{\{q(1+p)\}/\varpi}\,V^{\{p(1+q)\}/\varpi}\,dx)^{\varpi/\{(1+p)(1+q)\}}.$$

We now apply Schwarz's inequality again to the second integral on the right-hand side, taking this time

$$k = \frac{\varpi}{q(1+p)}, \quad \frac{k}{k-1} = \frac{\varpi}{p(1+q)};$$

and we obtain

$$\int uv\,dx \leq (\int UV\,dx)^{1/a}\,(\int U\,dx)^{q/(1+q)}\,(\int V\,dx)^{p/(1+p)},$$

which is equivalent to the result of the lemma. This proof fails if (*e.g.*) q is zero. But then the inequality becomes

$$(\int uv\,dx)^{1+p} \leq \int u^{1+p}\,v\,dx\,(\int v\,dx)^p$$

and is an immediate consequence of Schwarz's inequality.

In order to obtain Theorem (iii), we take $u = f(t)$ and $v = (x-t)^{\alpha-1}$, so that v^{1+q} is summable if

$$1 + q < \frac{1}{1-\alpha}.$$

Then

$$\left\{\int_0^x (x-t)^{\alpha-1} f(t)\,dt\right\}^{\{(1+p)(1+q)\}/(1-pq)} \leq \int_0^x (x-t)^{(1+q)(\alpha-1)} f^{1+p}\,dt$$
$$\times \left(\int_0^x f^{1+p}\,dt\right)^{\{q(1+p)\}/(1-pq)} \left\{\int_0^x (x-t)^{(1+q)(\alpha-1)}\,dt\right\}^{\{p(1+q)\}/(1-pq)}.$$

The two last factors on the right-hand side are continuous functions of x. The first is, apart from a constant factor, the integral of order $(1+q)(\alpha-1)+1>0$ of the summable function f^{1+p}, and is therefore summable, by Theorem (i). Hence the left-hand side is summable; and therefore $F_\alpha(x)$ has its ρ^{th} power summable if ρ is any number less than

$$\left(\frac{1+p}{1-\alpha}\right) \Big/ \left(1-\frac{p\alpha}{1-\alpha}\right) = 1 \Big/ \left(\frac{1}{1+p}-\alpha\right).$$

4. This proves Theorem (iii). Theorem (iv) is also very closely connected with a theorem of Prof. Young, the theorem that

$$\int_a^b g(x+t) f(t)\,dt$$

is a continuous function of x if f^{1+p} and $g^{(1+p)/p}$ are summable.* The theorem may be proved directly as follows. Taking $h>0$, we have

$$F_\alpha(x+h)-F_\alpha(x) = \int_0^x \{(x+h-t)^{\alpha-1}-(x-t)^{\alpha-1}\} f(t)\,dt + \int_x^{x+h} (x+h-t)^{\alpha-1} f(t)\,dt$$
$$= \phi_1 + \phi_2,$$

say. Now

$$|\phi_1| \leq \left(\int_0^x f^{1+p}\,dt\right)^{1/(1+p)} \left[\int_0^x \{(x-t)^{\alpha-1}-(x+h-t)^{\alpha-1}\}^{(1+p)/p}\,dt\right]^{p/(1+p)}.$$

But if u and v are positive, and $k>1$, we have†

$$u^k+v^k<(u+v)^k.$$

Hence

$$\{(x-t)^{\alpha-1}-(x+h-t)^{\alpha-1}\}^{(1+p)/p} \leq (x-t)^{\lambda-1}-(x+h-t)^{\lambda-1},$$

where

$$\lambda = 1-\frac{(1+p)(1-\alpha)}{p} > 0;$$

* W. H. Young, 'On a class of parametric integrals', *Proc. Roy. Soc.* (A), vol. 85, 1911, pp. 401—414.

† See, *e.g.*, J. L. W. V. Jensen, 'Sur les fonctions convexes et les inégalités entre les valeurs moyennes', *Acta Math.*, vol. 30, 1905, pp. 175—193 (p. 192).

and so

$$\int_0^x \{(x-t)^{\alpha-1} - (x+h-t)^{\alpha-1}\}^{(1+p)/p}\,dt$$
$$\leq \int_0^x \{(x-t)^{\lambda-1} - (x+h-t)^{\lambda-1}\}\,dt$$
$$= (1/\lambda)\,\{x^\lambda + h^\lambda - (x+h)^\lambda\},$$

which tends to zero with h. Thus ϕ_1 tends to zero with h.

On the other hand

$$|\phi_2| \leq \left(\int_x^{x+h} f^{1+p}\,dt\right)^{1/(1+p)} \left\{\int_x^{x+h} (x+h-t)^{\lambda-1}\,dt\right\}^{p/(1+p)}$$
$$= \lambda^{-p/(1+p)}\, h^{\lambda p/(1+p)} \left(\int_x^{x+h} f^{1+p}\,dt\right)^{1/(1+p)},$$

which plainly tends to zero with h. Thus $F_\alpha(x+h) - F_\alpha(x)$ tends to zero; and a similar argument applies to negative values of h. Hence $F_\alpha(x)$ is continuous.

5. Two supplementary questions require to be answered in order to complete Theorems (iii) and (iv). In the first place, can we take

$$\rho = 1 \Big/ \left(\frac{1}{1+p} - \alpha\right)$$

in Theorem (iii)? In the second place, can we take $\alpha = 1/(1+p)$ in Theorem (iv)? Or at any rate can we do this if we replace *continuous* in the conclusion by *bounded*?

The answer is in each case negative. In order to show this I consider the special case of (iii) in which $p=0$ and $\alpha = \frac{1}{2}$, and that of (iv) in which $p=1$. The suggested theorems would then run:

(*a*) *If f is summable, and* $F = \int_0^x \frac{f(t)\,dt}{\sqrt{(x-t)}}$, *then F^2 is summable*;

(*b*) *If f^2 is summable then F is bounded.*

I observe first that (*b*) implies (*a*). For suppose that (*b*) is true; and let h be any function whose square is summable. Then we have

$$\int_0^x Fh\,dt = \int_0^x h(t)\,dt \int_0^t \frac{f(u)\,du}{\sqrt{(t-u)}} = \int_0^x f(u)\,du \int_u^x \frac{h(t)\,dt}{\sqrt{(t-u)}},$$

if any one of these integrals has a meaning. But, if h^2 is summable and (*b*) is true, then

$$\int_u^x \frac{h(t)\,dt}{\sqrt{(t-u)}}$$

is a bounded function of u; and therefore its product by the summable function f is summable. It follows that Fh must be summable, for every h whose square is summable; and this is the necessary and sufficient condition that F^2 should be summable.* We thus deduce (a) from (b). In a very similar manner we could deduce (b) from (a), but this is not necessary for our present purpose.

It only remains, then, to show that (a) is false. I shall only indicate the proof, which may be effected in various ways. One method depends upon the theory of Fourier series. Suppose for simplicity that f is an even function, and that

$$f \sim \Sigma a_n \cos nx.$$

Then it is easy to show that

$$F \sim \Sigma (A_n \cos nx + B_n \sin nx),$$

where $$A_n \sim -\sqrt{\left(\frac{\pi}{2n}\right)} a_n, \quad B_n \sim \sqrt{\left(\frac{\pi}{2n}\right)} a_n;$$

and so, if F^2 is summable, the series

$$\Sigma \frac{|a_n|^2}{n}$$

is convergent. Our hypotheses therefore lead to the conclusion that this series is always convergent when $\Sigma a_n \cos nx$ is a Fourier series. And this is untrue, for

$$\Sigma \frac{\cos nx}{\sqrt{(\log n)}}$$

is a Fourier series,† that of a function continuous except for $x=0$, where it becomes infinite of order

$$|x|^{-1}\left(\log\frac{1}{|x|}\right)^{-\frac{3}{2}}.$$

Thus both (a) and (b) are false. The falsity of (a) may be proved more directly by taking

$$f(x) = x^{-1}\left(\log\frac{1}{x}\right)^{-\frac{3}{2}} \quad (x>0).$$

It will be found that

$$F(x) = \int_0^x t^{-1}\left(\log\frac{1}{t}\right)^{-\frac{3}{2}} \frac{dt}{\sqrt{(x-t)}} > \frac{K}{\sqrt{\{x\log(1/x)\}}}$$

for small values of x, so that F^2 is certainly not summable.

* F. Riesz, 'Untersuchungen über Systeme integrierbarer Funktionen', *Math. Annalen*, vol. 69, 1910, pp. 449–497 (p. 457).

† See W. H. Young, 'On the mode of oscillation of a Fourier series and of its allied series', *Proc. London Math. Soc.*, ser. 2, vol. 12, 1913, pp. 443–452.

APPROXIMATE FORMULÆ FOR LEGENDRE FUNCTIONS.

By *G. N. Watson.*

1. It is well known that in certain circumstances the limiting form of a Legendre function of high degree is expressible as a Bessel function. The formula which gives the limiting form of the Legendre function is due to Mehler,* and it may be written in the form

$$\lim_{n\to\infty} P_n(\cos\theta/n) = J_0(\theta).$$

Other formulæ of like nature, which need not be quoted here, connect the associated function P_n^m with the Bessel function of order m. The physical interpretation of these formulæ (derived from solutions of Laplace's equation) is that if we take the system of spheres, cones and coaxial planes which form the parametric surfaces of polar coordinates and which give rise to *spherical harmonics*, then these surfaces, in the immediate neighbourhood of the polar axis, are substantially of the same shape as the system of parallel planes, cylinders and coaxial planes which form the parametric surfaces of cylindrical coordinates and which give rise to *cylindrical harmonics*.

It is only natural that the 'method of steepest descents,' which yields the asymptotic expansions of many functions of large variables, and which is now known† to supply interesting approximate formulæ (in addition to complete asymptotic expansions), should supply formulæ of a type analogous to that obtained by Mehler.

The asymptotic expansions of Legendre functions of large degree, derived by the method of steepest descents, are discussed elsewhere;‡ in this paper I propose to discuss the approximate formulæ of Mehler's type; the results are of a more exact nature than previously-known formulæ, inasmuch as they give an upper limit to the difference between the Legendre function of high degree and a Bessel function of order zero.

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* *Crelle*, Bd. lxviii. (1868), p. 140.

† Cf. *Proc. Camb Phil. Soc.*, xix. (1917), pp. 96–110.

‡ See a forthcoming paper in the *Transactions of the Cambridge Philosophical Society*, xxii. (1918), pp. 277–308.

A typical formula, which will be established presently, is*

$$Q_n(z) = e^{-(n+\frac{1}{2})(\zeta-\tanh\zeta)}\sqrt{(\operatorname{sech}\zeta)}\,K_0\{(n+\tfrac{1}{2})\tanh\zeta\} + e^{-(n+\frac{1}{2})\zeta}\theta_1/(n+\tfrac{1}{2}),$$

where† $z=\cosh\zeta$ and $|\theta_1| < \frac{2}{3}|\sqrt{(\operatorname{sech}\zeta)}|$.

A formula of this character obviously has a wider range than formulæ of Mehler's type in which $n(z-1)^{\frac{1}{2}}$ tends a limit as $n\to\infty$, and z has to be taken to depend on n. The formula is also superior to formulæ of the type given by Laplace‡

$$P_n(\cos\theta) \sim \{\tfrac{1}{2}(n\pi\sin\theta)\}^{-\frac{1}{2}}\cos\{(n+\tfrac{1}{2})\theta - \tfrac{1}{4}\pi\},$$

which are definitely invalid when θ is nearly equal to 0 or π.

Moreover, the previously-known formulæ are all derivable without difficulty from the formulæ which are now to be given, provided that (in some cases) use is made of the Poisson-Lipschitz asymptotic expansion of Bessel functions of order zero.

2. We shall now obtain various definite integrals for Legendre functions by the aid of the contour integrals supplied by the method of steepest descents.

We take

$$I_1, I_2 = \int_{-1}^{1} (1-t)^n (1+t)^n (\cosh\zeta - t)^{-n-1}\,dt,$$

in which, in the case of I_1, the contour of integration is either a segment of the real axis or is reconcilable with it without crossing the point $t=\cosh\zeta$; while in the case of I_2, the contour passes above the point $t=\cosh\zeta$ when the imaginary part of $\cosh\zeta$ is positive and below it when the imaginary part of $\cosh\zeta$ is negative.

There is no particular difficulty in proving§ that

$$Q_n(\cosh\zeta) = I_1/2^{n+1},$$

$$P_n(\cosh\zeta) = \pm(I_2 - I_1)/(2^{n+1}\pi i),$$

the upper or lower sign being taken according as $I(\cosh\zeta) \gtrless 0$.

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* The function K_0 is the Bessel function 'of imaginary argument' introduced by Basset; it may be defined by the integral

$$K_0(z) = \int_0^\infty e^{-z\cosh t}\,dt.$$

† To determine ζ uniquely when z is given, we suppose that $\zeta = \xi + i\eta$, where ξ, η are real and $\xi \geq 0$, $-\pi \leq \eta \leq \pi$.

‡ *Mécanique Céleste*, v. (1823), livre 11; supplément 1.

§ Cf. Hobson, *Phil. Trans.*, 187 A (1896), pp. 443–531; and Whittaker and Watson, *Modern Analysis*, chap. xv.

The method of steepest descents requires the choice of contours on which the imaginary part of

$$\log \frac{\cosh \zeta - t}{1 - t^2}$$

is constant, the contours passing through the stationary points of this function of t.

Now
$$\frac{d}{dt} \log \frac{\cosh \zeta - t}{1 - t^2}$$

vanishes when $t = e^{\pm \zeta}$; and it may be shewn* that the contour through the point $e^{-\zeta}$ is an appropriate contour for the integral I_1, while the contour through e^{ζ} is appropriate for I_2. The contours are (in each case) nodal cubics.

Taking the integral I_2 first, we write

$$\tau = \log \frac{t - \cosh \zeta}{t^2 - 1} + \log 2e^{\zeta},$$

so that, as t passes from -1 to e^{ζ} and thence to 1, τ decreases from $+\infty$ to 0 and then increases to $+\infty$, the value of τ being real throughout.

We thus get

$$I_2 = \left\{ \int_{\infty}^{0} + \int_{0}^{\infty} \right\} (\cosh \zeta - t)^{-1} 2^n e^{n\zeta} e^{-n\tau} (dt/d\tau)\, d\tau,$$

and
$$\frac{dt}{d\tau} = -\frac{(t^2 - 1)(t - \cosh \zeta)}{t^2 - 2t \cosh \zeta + 1}.$$

Let t_1, t_2 be the values of t corresponding to any assigned positive value of τ, of which t_1 is on that part of the contour which lies between the stationary point and $+1$.

Then, since t_1, t_2 are the roots of the quadratic

$$t^2 - 1 = 2e^{\zeta - \tau}(t - \cosh \zeta),$$

we have
$$\cosh \zeta = (1 + t_1 t_2)/(t_1 + t_2),$$

and

$$\frac{dt_1/d\tau}{\cosh \zeta - t_1} - \frac{dt_2/d\tau}{\cosh \zeta - t_2} = \frac{t_1^2 - 1}{t_1^2 - 2t_1 \cosh \zeta + 1} - \frac{t_2^2 - 1}{t_2^2 - 2t_2 \cosh \zeta + 1}$$

$$= \frac{t_1 + t_2}{t_1 - t_2} - \frac{t_2 + t_1}{t_2 - t_1} = \frac{4e^{\zeta - \tau}}{t_1 - t_2}.$$

* Details of the analysis are given in the forthcoming paper to which reference has already been made.

Now $$t_1,\ t_2 = e^{\zeta - t} \pm (1 - e^{-\tau})^{\frac{1}{2}} (1 - e^{2\zeta - t})^{\frac{1}{2}},$$

where the upper sign refers to t_1 if $\arg (1 - e^{2\zeta - t}) \to 0$ as $\tau \to \infty$. Hence

$$I_2 = 2^{n+1} e^{(n+1)\zeta} \int_0^\infty \frac{e^{-(n+1)\tau}\, d\tau}{(1 - e^{-\tau})^{\frac{1}{2}} (1 - e^{2\zeta - \tau})^{\frac{1}{2}}},$$

provided that* η is not 0 or $\pm \pi$.

In like manner, with the same limitation,*

$$I_1 = 2^{n+1} e^{-(n+1)\zeta} \int_0^\infty \frac{e^{-(n+1)\tau}\, d\tau}{(1 - e^{-\tau})^{\frac{1}{2}} (1 - e^{-2\zeta - \tau})^{\frac{1}{2}}}.$$

By an elementary transformation we now easily obtain the integrals

$$Q_n(z) = e^{-(n+\frac{1}{2})\zeta} \int_0^\infty \frac{e^{-(n+1)\tau} d\tau}{(1 - e^{-\tau})^{\frac{1}{2}}} \{(1 + e^{-\tau}) \sinh \zeta + (1 - e^{-\tau}) \cosh \zeta\}^{-\frac{1}{2}},$$

$$P_n(z) = \frac{e^{(n+\frac{1}{2})\zeta}}{\pi} \int_0^\infty \frac{e^{-(n+1)\tau} d\tau}{(1 - e^{-\tau})^{\frac{1}{2}}} \{(1 + e^{-\tau}) \sinh \zeta - (1 - e^{-\tau}) \cosh \zeta\}^{-\frac{1}{2}}$$

$$+ \frac{e^{-(n+\frac{1}{2})\zeta \pm \frac{1}{2}\pi i}}{\pi} \int_0^\infty \frac{e^{-(n+1)\tau} d\tau}{(1 - e^{-\tau})^{\frac{1}{2}}} \{(1 + e^{-\tau}) \sinh \zeta + (1 - e^{-\tau}) \cosh \zeta\}^{-\frac{1}{2}}$$

where, as usual, $z = \cosh \zeta$, and the upper or lower sign is taken according as η is positive or negative.

3. We shall now take the integrals just obtained in § 2 and obtain expressions which differ from their integrands by functions of τ which are negligible in comparison with the integrands for sufficiently small values of τ; these approximations are as follows:

$$\frac{e^{-\frac{1}{2}\tau}}{(1 - e^{-\tau})^{\frac{1}{2}}} \{(1 + e)^{-\tau} \sinh \zeta \pm (1 - e^{-\tau}) \cosh \zeta\}^{-\frac{1}{2}}$$

$$\sim \frac{\{\sinh \zeta \pm \frac{1}{2}\tau \cosh \zeta\}^{-\frac{1}{2}}}{\sqrt{(2\tau)}}.$$

We proceed to obtain the requisite upper limit for the modulus of the difference of these approximately equal functions.

Let μ be any positive number (including zero), and let the lower bound of $|1 + \mu \tanh \zeta|$, as μ varies from 0 to $+\infty$,

* Allowance can be made for the exceptional cases by a suitable indentation of the contour if the mode of approach of $\cosh \zeta$ to the real axis is specified.

be Ω_1; so that Ω_1 is a positive number which depends on ζ but not on μ, and

$$|1+\mu\tanh\zeta|\geq\Omega_1.$$

As μ varies from 0 to $+\infty$, the point $1+\mu\tanh\zeta$ traces out a ray in the Argand diagram, and so, if $\mu'>0$, we have

$$|\arg(1+\mu\tanh\zeta)-\arg(1+\mu'\tanh\zeta)|<\pi,$$

if $\arg(1+\mu\tanh\zeta)$ varies continuously with μ.

Next, we observe that

$$\frac{1}{\sqrt{U}}-\frac{1}{\sqrt{V}}=\frac{V-U}{\sqrt{(UV)}\{\sqrt{U}+\sqrt{V}\}};$$

and, if $\arg U$ differs from $\arg V$ by less than π, $|\sqrt{U}+\sqrt{V}|$ exceeds both $|\sqrt{U}|$ and $|\sqrt{V}|$.

Hence, obviously, when $\tau\geq 0$,

$$\begin{aligned}&|\{(1+e^{-\tau})\sinh\zeta+(1-e^{-\tau})\cosh\zeta\}^{-\frac{1}{2}}\\&\qquad-\{(1+e^{-\tau})\sinh\zeta+\tfrac{1}{2}\tau(1+e^{-\tau})\cosh\zeta\}^{-\frac{1}{2}}|\\&\leq\frac{\{\frac{1}{2}\tau(1+e^{-\tau})-(1-e^{-\tau})\}|\cosh\zeta|}{|\cosh\zeta|^{\frac{3}{2}}(1-e^{-\tau})\{\frac{1}{2}\tau(1+e^{-\tau})\}^{\frac{1}{2}}|\{1+\tanh\zeta\coth\frac{1}{2}\tau\}\{1+2\tau^{-1}\tanh\zeta\}^{\frac{1}{2}}|}\\&=\{\tfrac{1}{2}\tau\coth\tfrac{1}{2}\tau-1\}|\cosh\zeta|^{-\frac{1}{2}}\Omega_1^{-\frac{3}{2}}\{\tfrac{1}{2}\tau(1+e^{-\tau})\}^{-\frac{1}{2}}.\end{aligned}$$

Consequently

$$\begin{aligned}&\left|\frac{e^{-(n+1)\tau}}{(1-e^{-\tau})^{\frac{1}{2}}}\left[\{(1+e^{-\tau})\sinh\zeta+(1-e^{-\tau})\cosh\zeta\}^{-\frac{1}{2}}\right.\right.\\&\qquad\left.\left.-\{(1+e^{-\tau})\sinh\zeta+\tfrac{1}{2}\tau(1+e^{-\tau})\cosh\zeta\}^{-\frac{1}{2}}\right]\right|\\&\leq e^{-(n+1)\tau}\{\tfrac{1}{2}\tau\coth\tfrac{1}{2}\tau-1\}|\sqrt{\operatorname{sech}\zeta}|\,\Omega_1^{-\frac{3}{2}}\{\tfrac{1}{2}\tau(1-e^{-2\tau})\}^{-\frac{1}{2}}\\&\leq\tfrac{1}{2}e^{-(n+\frac{1}{2})\tau}|\sqrt{\operatorname{sech}\zeta}|\,\Omega_1^{-\frac{3}{2}},\end{aligned}$$

since $\coth\frac{1}{2}\tau-(2/\tau)$, being an increasing function of τ, is less than 1, its limit when τ is infinite; and $\tau^{-1}\sinh\tau\geq 1$.

Next, we have

$$\begin{aligned}(2\tau e^{-\tau})^{-\frac{1}{2}}-(1-e^{-2\tau})^{-\frac{1}{2}}&=\frac{1-e^{-2\tau}-2\tau e^{-\tau}}{\{2\tau e^{-\tau}(1-e^{-2\tau})\}^{\frac{1}{2}}\{(1-e^{-2\tau})^{\frac{1}{2}}+(2\tau e^{-\tau})^{\frac{1}{2}}\}}\\&<\frac{1-e^{-2\tau}-2\tau e^{-\tau}}{2(2\tau e^{-\tau})^{\frac{1}{2}}(1-e^{-2\tau})}=\frac{\sinh\tau-\tau}{(2\tau e^{-\tau})^{\frac{1}{2}}\sinh\tau}<\tfrac{1}{6}\tau^{\frac{3}{2}}e^{\frac{1}{2}\tau}/\sqrt{2},\end{aligned}$$

since $1-e^{-2\tau}>2\tau e^{-\tau}$ and since $\sinh\tau-\tau<\frac{1}{6}\tau^2\sinh\tau$, by expansion in series. Consequently

$$\left| \frac{e^{-(n+1)\tau}}{(1-e^{-\tau})^{\frac{1}{2}}} \{(1+e^{-\tau})\sinh\zeta + (1-e^{-\tau})\cosh\zeta\}^{-\frac{1}{2}} \right.$$
$$\left. - \frac{e^{-(n+1)\tau}}{(2\tau e^{-\tau})^{\frac{1}{2}}} \{\sinh\zeta + \tfrac{1}{2}\tau\cosh\zeta\}^{-\frac{1}{2}} \right|$$
$$< \tfrac{1}{2} e^{-(n+\frac{1}{2})\tau} |\sqrt{\operatorname{sech}\zeta}|\, \Omega_1^{-\frac{3}{2}} + \tfrac{1}{6}\tau e^{-(n+\frac{1}{2})\tau} |\sqrt{\operatorname{sech}\zeta}|\, \Omega_1^{-\frac{1}{2}};$$

and therefore

$$\int_0^\infty \frac{e^{-(n+1)\tau}\,d\tau}{(1-e^{-\tau})^{\frac{1}{2}}} \{(1+e^{-\tau})\sinh\zeta + (1-e^{-\tau})\cosh\zeta\}^{-\frac{1}{2}}$$
$$= \int_0^\infty \frac{e^{-(n+\frac{1}{2})\tau}\,d\tau}{\sqrt{\{2\tau(\sinh\zeta + \frac{1}{2}\tau\cosh\zeta)\}}} + \frac{\theta_1}{n+\frac{1}{2}},$$

where

$$|\theta_1| \leq (n+\tfrac{1}{2})\,\Omega_1^{-\frac{1}{2}} |\sqrt{\operatorname{sech}\zeta}| \int_0^\infty (\tfrac{1}{2}\Omega_1^{-1} + \tfrac{1}{6}\tau) e^{-(n+\frac{1}{2})\tau}\,d\tau$$
$$= \Omega_1^{-\frac{1}{2}} |\sqrt{\operatorname{sech}\zeta}| \{\tfrac{1}{2}\Omega_1^{-1} + \tfrac{1}{6}(n+\tfrac{1}{2})^{-1}\}.$$

In like manner, if Ω_2 be the lower bound of $|1-\mu\tanh\zeta|$, we have

$$\int_0^\infty \frac{e^{-(n+1)\tau}\,d\tau}{(1-e^{-\tau})^{\frac{1}{2}}} \{(1+e^{-\tau})^{\frac{1}{2}}\sinh\zeta - (1-e^{-\tau})\cosh\zeta\}^{-\frac{1}{2}}$$
$$= \int_0^\infty \frac{e^{-(n+\frac{1}{2})\tau}\,d\tau}{\sqrt{\{2\tau(\sinh\zeta - \frac{1}{2}\tau\cosh\zeta)\}}} + \frac{\theta_2}{n+\frac{1}{2}},$$

where $|\theta_2| \leq \Omega_2^{-\frac{1}{2}} |\sqrt{\operatorname{sech}\zeta}| \{\frac{1}{2}\Omega_2^{-1} + \frac{1}{6}(n+\frac{1}{2})^{-1}\}$.

4. The integrals just obtained, which are approximations to the original integrals when n is large, are easily expressible in terms of Bessel functions. Thus*

$$\int_0^\infty \frac{e^{-(n+\frac{1}{2})\tau}\,d\tau}{\sqrt{\{2\tau(\sinh\zeta + \frac{1}{2}\tau\cosh\zeta)\}}} = \frac{e^{(n+\frac{1}{2})\tanh\zeta}}{\sqrt{\cosh\zeta}} K_0\{(n+\tfrac{1}{2})\tanh\zeta\},$$

and so we have

$$I_1 = 2^{n+1}\sqrt{(\operatorname{sech}\zeta)}\, e^{-(n+\frac{1}{2})(\zeta-\tanh\zeta)}$$
$$\times K_0\{(n+\tfrac{1}{2})\tanh\zeta\} + 2^{n+1} e^{-(n+\frac{1}{2})\zeta}\theta_1/(n+\tfrac{1}{2}).$$

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* The function $K_m(x)$ is the solution of Bessel's associated equation

$$x^2y'' + xy' - (x^2+m^2)y = 0,$$

which possesses the asymptotic expansion

$$K_m(x) \sim \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x}\cos m\pi \left[1 + \sum_{r=1}^{\infty} \frac{\{4m^2-1^2\}\{4m^2-3^2\}\ldots\{4m^2-(2r-1)^2\}}{r!\,2^{3r}x^r}\right],$$

when $|\arg x| < \frac{3}{2}\pi$; it is also defined as (the limit of) $\frac{1}{2}\pi\cot m\pi\{I_{-m}(x) - I_m(x)\}$; see Gray and Mathews, *Bessel Functions*, p. 68; Whittaker and Watson, *Modern Analysis*, pp. 367, 377; and Hankel, *Math. Ann.*, I. (1869), p. 498.

Since* $R(\tanh\zeta) \geq 0$, it is evident that $\Omega_1 = 1$, and so, n being positive, we have $|\theta_1| < \frac{2}{3}|\sqrt{(\operatorname{sech}\zeta)}|$.

The expression for $Q_n(z)$ quoted at the beginning of the section follows at once from this formula.

It is evident that n may be allowed to assume complex values, provided that $R(n) > 0$; we then have to modify the remainder term slightly by writing $R(n+\frac{1}{2})$ for $n+\frac{1}{2}$ in the denominator; the approximation fails when arg n approaches $\pm\frac{1}{2}\pi$.

When $-1 < z < 1$, we define $Q_n(z)$ to be the mean of the values on each side of the cut; writing $z = \cos\eta$ where $0 < \eta < \pi$, we see that

$$Q_n(\cos\eta) = \tfrac{1}{2}e^{-(n+\frac{1}{2})(\eta-\tan\eta)i}\sqrt{\operatorname{sech}(0+i\eta)}\, K_0\{(n+\tfrac{1}{2})e^{\frac{1}{2}\pi i}\tan\eta\}$$
$$+ \tfrac{1}{2}e^{(n+\frac{1}{2})(\eta-\tan\eta)i}\sqrt{\operatorname{sech}(0-i\eta)}\, K_0\{(n+\tfrac{1}{2})e^{-\frac{1}{2}\pi i}\tan\eta\}$$
$$+ \theta_3/(n+\tfrac{1}{2}),$$

where $|\theta_3| < \frac{2}{3}|\sqrt{(\sec\eta)}|$.

When η is a positive *acute* angle, this gives†

$$Q_n(\cos\eta) = \pi\sin\{(n+\tfrac{1}{2})(\tan\eta-\eta)\}\sqrt{(\sec\eta)}\, J_0\{(n+\tfrac{1}{2})\tan\eta\}$$
$$-\cos\{(n+\tfrac{1}{2})(\tan\eta-\eta)\}\sqrt{(\sec\eta)}\, Y_0\{(n+\tfrac{1}{2})\tan\eta\} + \theta_3/(n+\tfrac{1}{2}),$$

where $|\theta_3| < \frac{2}{3}\sqrt{\sec\eta}$.

5. The corresponding formula for $P_n(z)$ is rather more complicated. In the first place, Ω_2 is as small as the shortest distance from the origin of the line traced out by the point $1+\mu\tanh\zeta$ as μ varies from $-\infty$ to $+\infty$. Hence we have

$$\Omega_2 = |\sin 2\eta|\,\{\sin^2 2\eta + \sinh^2 2\xi\}^{-\frac{1}{2}}.$$

Next we have to evaluate the integrals

$$\int_0^\infty \frac{e^{-(n+\frac{1}{2})\tau}\,d\tau}{\sqrt{\{2\tau(\sinh\zeta \pm \frac{1}{2}\tau\cosh\zeta)\}}}.$$

These are easily expressed in terms of the Hankel-Nielsen‡ functions $H_n^{(1)}$ and $H_n^{(2)}$; we have

$$H_0^{(1)}(x) = (2/x)^{\frac{1}{2}}\pi^{-1}e^{i(x-\frac{1}{4}\pi)}\int_0^\infty e^{-t}(1+\tfrac{1}{2}it/x)^{-\frac{1}{2}}t^{-\frac{1}{2}}\,dt,$$

$$H_0^{(2)}(x) = (2/x)^{\frac{1}{2}}\pi^{-1}e^{-i(x-\frac{1}{4}\pi)}\int_0^\infty e^{-t}(1-\tfrac{1}{2}it/x)^{-\frac{1}{2}}t^{-\frac{1}{2}}\,dt,$$

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* We have $R(\tanh\zeta) = \sinh 2\xi/(\cosh 2\xi + \cos 2\eta)$.

† When m is made an integer after differentiation, $Y_m(x)$ may be defined as

$$\frac{\partial J_m(x)}{\partial m} - (-)^m\frac{\partial J_{-m}(x)}{\partial m}.$$

It is then the solution of Bessel's equation of which the dominant term in the asymptotic expansion is $Y_m(x) \sim (2\pi/x)^{\frac{1}{2}}\sin(x-\frac{1}{2}m\pi-\frac{1}{4}\pi)$.

‡ Nielsen, *Handbuch der Cylinderfunktionen*, p. 153.

provided that $|\arg x| < \frac{1}{2}\pi$; further

$$H_0^{(1)}(x) = J_0(x) + i\pi^{-1} Y_0(x),$$
$$H_0^{(2)}(x) = J_0(x) - i\pi^{-1} Y_0(x).$$

Hence it follows that, when $-\frac{1}{2}\pi < \eta < \frac{1}{2}\pi$, we have

$$\int_0^\infty \frac{e^{-(n+\frac{1}{2})\tau}\, d\tau}{\sqrt{\{2\tau(\sinh\zeta \pm \frac{1}{2}\tau\cosh\zeta)\}}} = \tfrac{1}{2}\pi\sqrt{(\operatorname{sech}\zeta)}$$
$$\times e^{\mp(n+\frac{1}{2})\tanh\zeta}\, e^{\frac{1}{4}\pi i(1\mp 1)} H_0^{(1)}\{\mp(n+\tfrac{1}{2})\, i\tanh\zeta\}$$

$$\int_0^\infty \frac{e^{-(n+\frac{1}{2})\tau}\, d\tau}{\sqrt{\{2\tau(\sinh\zeta \pm \frac{1}{2}\tau\cosh\zeta)\}}} = \tfrac{1}{2}\pi\sqrt{(\operatorname{sech}\zeta)}$$
$$\times e^{\pm(n+\frac{1}{2})\tanh\zeta}\, e^{\frac{1}{4}\pi i(-1\mp 1)} H_0^{(2)}\{\mp(n+\tfrac{1}{2})\, i\tanh\zeta\},$$

upper or lower signs being taken according as η is $\pm$. That is to say

$$\int_0^\infty \frac{e^{-(n+\frac{1}{2})\tau}\, d\tau}{\sqrt{\{2\tau(\sinh\zeta + \frac{1}{2}\tau\cosh\zeta)\}}} = \tfrac{1}{2}\sqrt{(\operatorname{sech}\zeta)}\, e^{(n+\frac{1}{2})\tanh\zeta}\{\mp i J_0 \mp i Y_0\},$$

$$\int_0^\infty \frac{e^{-(n+\frac{1}{2})\tau}\, d\tau}{\sqrt{\{2\tau(\sinh\zeta - \frac{1}{2}\tau\cosh\zeta)\}}} = \tfrac{1}{2}\sqrt{(\operatorname{sech}\zeta)}\, e^{-(n+\frac{1}{2})\tanh\zeta}\{\pi J_0 \pm i Y_0\},$$

the arguments of the Bessel functions being $\mp(n+\frac{1}{2})\, i\tanh\zeta$. Consequently *

$$P_n(z) = \cosh\{(n+\tfrac{1}{2})(\tanh\zeta - \zeta)\}\sqrt{(\operatorname{sech}\zeta)}\, J_0\{\mp(n+\tfrac{1}{2})\, i\tanh\zeta\}$$
$$\mp i\pi^{-1}\sinh\{(n+\tfrac{1}{2})(\tanh\zeta - \zeta)\}\sqrt{(\operatorname{sech}\zeta)}\, Y_0\{\mp(n+\tfrac{1}{2})\, i\tanh\zeta\}$$
$$+ \pi^{-1}\{e^{(n+\frac{1}{2})\zeta}\theta_2 \pm i e^{-(n+\frac{1}{2})\zeta}\theta_1\}/(n+\tfrac{1}{2}).$$

In particular, when η is a positive acute angle and $\xi = 0$, we have $\Omega_1 = \Omega_2 = 1$, and so

$$P_n(\cos\eta) = \cos\{(n+\tfrac{1}{2})(\tan\eta - \eta)\}\sqrt{(\sec\eta)}\, J_0\{(n+\tfrac{1}{2})\tan\eta\}$$
$$+ \pi^{-1}\sin\{(n+\tfrac{1}{2})(\tan\eta - \eta)\}\sqrt{(\sec\eta)}\, Y_0\{(n+\tfrac{1}{2})\tan\eta\} + \theta_4/(n+\tfrac{1}{2}),$$

where $|\theta_4| < \frac{4}{3}\pi^{-1}\sqrt{(\sec\eta)}$.

6. The formula just obtained for $P_n(z)$ is obviously valueless when Ω_2^{-1} is large, *i.e.* when $\sin 2\eta\operatorname{cosech} 2\xi$ is small; if A be a small positive constant, the curves on which $|\sin 2\eta\operatorname{cosech} 2\xi| = A$ in the z plane are nearly coincident with the rays $\arg(z-1) = \pm 2\tan^{-1}A$ when $|z-1|$ is

* The discontinuity on the real axis between the points $z = \pm 1$ is only apparent, in view of the discontinuity of η.

small; and consequently we need an approximation to $P_n(z)$ when $|\arg(z-1)| \leq 2\tan^{-1}A$ and $|z-1|$ is small. We may suppose A chosen so that we have $|\zeta| \leq 1$ throughout this region; and so, if we can obtain an approximation to $P_n(z)$ when $|\zeta| \leq 1$, this, combined with formulæ already obtained, will suffice to give approximate formulæ for $P_n(z)$ throughout the plane. When $z-1$ is *positive*, the contour supplied by the method of steepest descents is a circle and a straight line; and the circle is that which arises in deriving Laplace's integral from Schläfli's contour integral.*

We therefore take Laplace's integral

$$P_n(z) = \frac{1}{\pi}\int_0^{\pi} \{z + (z^2-1)^{\frac{1}{2}}\cos\phi\}^{-n-1}\, d\phi,$$

and make the obvious substitution

$$z + (z^2+1)^{\frac{1}{2}}\cos\phi = e^{\tau},$$

which gives

$$P_n(z) = \frac{1}{\pi}\int_{-\zeta}^{\zeta} e^{-n\tau}\{(1-e^{\tau-\zeta})(e^{\tau+\zeta}-1)\}^{-\frac{1}{2}}\, d\tau.$$

This result, proved when $z-1>0$, holds by the ordinary theory of analytic continuation when $R(z) \geq 0$; and so it certainly holds throughout the region for which $|\zeta| \leq 1$.

Now consider

$$\{\zeta^{-1}\sinh\zeta(\zeta^2-\tau^2)\}^{-\frac{1}{2}} - \{2(\cosh\zeta-\cosh\tau)\}^{-\frac{1}{2}}$$

$$= [2(\cosh\zeta-\cosh\tau) - \zeta^{-1}\sinh\zeta(\zeta^2-\tau^2)]$$

$$\times \{2(\cosh\zeta-\cosh\tau)\}^{\frac{1}{2}}\{\zeta^{-1}\sinh\zeta(\zeta^2-\tau^2)\}^{-\frac{1}{2}}$$

$$\times [\{2(\cosh\zeta-\cosh\tau)\}^{\frac{1}{2}} + \{\zeta^{-1}\sinh\zeta(\zeta^2-\tau^2)\}^{\frac{1}{2}}]^{-1}.$$

We have

$$|[\{2(\cosh\zeta-\cosh\tau) - \zeta^{-1}\sinh\zeta(\zeta^2-\tau^2)\}(\zeta^2-\tau^2)^{-2}]|$$

$$= |\sum_{m=0}^{\infty} \tfrac{1}{2}(2m!)\{\tau^{2m-4} + 2\tau^{2m-6}\zeta^2 + 3\tau^{2m-8}\zeta^4 + \ldots + (m-1)\zeta^{2m-4}\}|$$

$$\leq \sum_{m=0}^{\infty} m(m-1)|\zeta|^{2m-4}/(2m)! \leq \sum_{m=2}^{\infty} m(m-1)/(2m)!$$

$$< \tfrac{1}{12}\sum_{m=2}^{\infty} 10^{2-m} < \tfrac{1}{10}.$$

* This is the only case known in which a curvilinear contour supplied by the method of steepest descents is one which has been previously employed in analysis. Details of the use of this circular contour (the introduction of which is due to Mr. Hodgkinson) are given by Whittaker and Watson, *Modern Analysis*, p. 306.

Similarly

$$2(\cosh\zeta-\cosh\tau)(\zeta^2-\tau^2)^{-1}=1-\theta_5\sum_{m=2}^{\infty}|\zeta^{2m-2}|/(2m-1)!$$

where $|\theta_5|<1$, and so it is equal to $1-\frac{1}{4}\theta_6$, where $|\theta_6|$ is certainly less than 1.

From this result it is evident that

$$|\{\zeta^{-1}\sinh\zeta(\zeta^2-\tau^2)\}^{-\frac{1}{2}}-\{2(\cosh\zeta-\cosh\tau)\}^{-\frac{1}{2}}|<\theta_7|(\zeta^2-\tau^2)|^{\frac{1}{2}},$$

where $|\theta_7|<\frac{1}{5}.6^{-\frac{3}{2}}$.

Now

$$\int_{-}^{\zeta}e^{-n\tau}|(\zeta^2-\tau^2)|^{\frac{1}{2}}d\tau\Big|\leq|\zeta|^2\int_{-1}^{1}e^{-\lambda nR(\zeta)}d\lambda$$

$$=2|\zeta|^3(n\xi)^{-1}\sinh n\xi.$$

Also

$$\int_{-\zeta}^{\zeta}e^{-n\tau}(\zeta^2-\tau^2)^{-\frac{1}{2}}d\tau=\pi I_0(n\zeta).$$

Consequently, when $|\zeta|<1$,

$$P_n(z)=(\zeta\operatorname{cosech}\zeta)^{\frac{1}{2}}I_0(n\zeta)+\theta_8/n,$$

where

$$|\theta_8|<\tfrac{2}{5}.6^{-\frac{3}{2}}|\zeta|^2(n\xi)^{-1}\sinh n\xi.$$

The term involving the Bessel function gives a good approximation when n is large and $\zeta\xi^{-1}$ is not large, *i.e.*, when z is in the neighbourhood of the part of the real axis on the right of $z=1$.

In particular, when $0<\xi<1$,

$$P_n(\cosh\xi)=(\xi\operatorname{cosech}\xi)^{\frac{1}{2}}I_0(n\xi)+\theta_8/n,$$

where

$$|\theta_8|<\tfrac{2}{5}.6^{-\frac{3}{2}}(\xi/n)\sinh n\xi.$$

It is interesting to compare these results with Macdonald's formal expansion *

$$P_n^{-m}(\cos\theta)=(n+\tfrac{1}{2})^{-m}(\cos\tfrac{1}{2}\theta)^m\left[J_m(x)+\sin^2\tfrac{1}{2}\theta\{\tfrac{1}{6}xJ_{m+3}(x)\right.$$

$$\left.-(m+1)J_{m+2}(x)+2x^{-1}(m+\tfrac{1}{2})^2J_{m+1}(x)\}+\ldots\right],$$

where $x=2n\sin^2\frac{1}{2}\theta$.

It does not appear practicable to extend these results to Legendre functions of any order. The contour integrals supplied by the method of steepest descents are simple, but the definite integrals which are derived from them appear to be somewhat intractable.

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SOME DIFFERENTIAL EQUATIONS OCCURRING IN THE ELECTRICAL THEORY OF RADIATION.

By *Dr. H. Bateman.*

1. It is known that in the case of an electromagnetic field in free æther the energy can be regarded as travelling with the velocity of light only when the electric intensity $\mathbf{E}$ and the magnetic intensity $\mathbf{H}$ are connected by the equations

$$E^2 - H^2 = 0, \quad (\mathbf{E}\,\mathbf{H}) = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

which signify that $\mathbf{E}$ and $\mathbf{H}$ are perpendicular and equal in magnitude. If we write $\mathbf{M} = \mathbf{H} + i\mathbf{E}$, the problem of finding all the fields that possess the property just mentioned at each point and at each instant is equivalent to that of finding all the solutions of the equations

$$c \operatorname{rot} \mathbf{M} + i \frac{\partial \mathbf{M}}{\partial t} = 0, \quad \operatorname{div} \mathbf{M} = 0 \ldots\ldots\ldots\ldots (2)$$

that satisfy the condition $M^2 = 0$.

Let θ be a homogeneous function of degree zero of the three components of $\mathbf{M}$, then

$$\theta = F(M_x,\ M_y,\ M_z) \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where

$$M_x \frac{\partial F}{\partial M_x} + M_y \frac{\partial F}{\partial M_y} + M_z \frac{\partial F}{\partial M_z} = 0 \ldots\ldots\ldots\ldots (4).$$

We shall now show that this quantity θ satisfies the partial differential equations*

$$c[\nabla\theta,\ \mathbf{M}] + i\mathbf{M} \frac{\partial \theta}{\partial t} = 0, \quad (\nabla\theta,\ \mathbf{M}) = 0 \ldots\ldots\ldots (5).$$

Since

$$\nabla\theta = \frac{\partial F}{\partial M_x} \nabla M_x + \frac{\partial F}{\partial M_y} \nabla M_y + \frac{\partial F}{\partial M_z} \nabla M_z$$

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* We shall use the symbol $\nabla\theta$ to denote the vector whose components are $\frac{\partial\theta}{\partial x}$, $\frac{\partial\theta}{\partial y}$, $\frac{\partial\theta}{\partial z}$ and the symbols $(\mathbf{E}\,\mathbf{H})$, $[\mathbf{E}\,\mathbf{H}]$ for the scalar and vector products of two vectors $\mathbf{E}$ and $\mathbf{H}$.

we have to prove that

$$\left.\begin{aligned} c[\nabla M_x,\ \mathbf{M}] + i\mathbf{M}\frac{\partial M_x}{\partial t} &= \mathbf{K}M_x \\ (\nabla M_x,\ \mathbf{M}) &= JM_x \end{aligned}\right\} \ldots\ldots\ldots\ldots (6),$$

with two similar pairs of equations involving the y and z components of $\mathbf{M}$ and the same multipliers $\mathbf{K}$ and J. The quantities on the left-hand sides of these equations must have the form indicated on the right in order that the equations (5) may be satisfied in consequence of (4).

Now we have the identity

$$[\nabla M_x,\ \mathbf{M}] = \mathbf{M}\left(\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}\right) + M_x\nabla M_y - M_y\nabla M_z + \mathbf{I},$$

where $\mathbf{I}$ denotes the vector whose components are

$$-(\mathbf{M}\,\mathrm{rot}\,\mathbf{M}),\ \left(\mathbf{M}\frac{\partial \mathbf{M}}{\partial z}\right) - M_z\,\mathrm{div}\,\mathbf{M} \text{ and } M_y\,\mathrm{div}\,\mathbf{M} - \left(\mathbf{M}\frac{\partial \mathbf{M}}{\partial y}\right)$$

respectively. This vector $\mathbf{I}$ is easily seen to be zero on account of equations (2) and the relation $M' = 0$ which gives

$$\left(\mathbf{M}\frac{\partial \mathbf{M}}{\partial y}\right) = 0,\ \left(\mathbf{M}\frac{\partial \mathbf{M}}{\partial z}\right) = 0,\ \left(\mathbf{M}\frac{\partial \mathbf{M}}{\partial t}\right) = 0,\ \left(\mathbf{M}\frac{\partial \mathbf{M}}{\partial x}\right) = 0.$$

The relations $M' = 0$, $\left(\mathbf{M}\dfrac{\partial \mathbf{M}}{\partial x}\right) = 0$, also indicate that

$$\left[\mathbf{M}\frac{\partial \mathbf{M}}{\partial x}\right] = \Gamma_x\mathbf{M},$$

where Γ_x is some scalar factor. In a similar way we find that

$$\left[\mathbf{M}\frac{\partial \mathbf{M}}{\partial y}\right] = \Gamma_y\mathbf{M},\quad \left[\mathbf{M}\frac{\partial \mathbf{M}}{\partial z}\right] = \Gamma_z\mathbf{M},\quad \left[\mathbf{M}\frac{\partial \mathbf{M}}{\partial t}\right] = \Gamma_t\mathbf{M};$$

hence we may write

$$M_y\nabla M_z - M_z\nabla M_y = \Gamma M_x,$$

where Γ denotes the vector with components Γ_x, Γ_y, Γ_z. Making use of (2), we find eventually that

$$c[\nabla M_x,\ \mathbf{M}] + i\mathbf{M}\frac{\partial M_x}{\partial t} = -c\Gamma M_x,$$

which agrees with the first of equations (6) if $\mathbf{K} = -c\Gamma$.

Again, we have

$$(\nabla M_x,\ \mathbf{M}) = \left(\mathbf{M}\frac{\partial \mathbf{M}}{\partial x}\right) + M_y\left(\frac{\partial M_x}{\partial y} - \frac{\partial M_y}{\partial x}\right) - M_z\left(\frac{\partial M_z}{\partial x} - \frac{\partial M_x}{\partial z}\right)$$

$$= \frac{i}{c}\left(M_y\frac{\partial M_z}{\partial t} - M_z\frac{\partial M_y}{\partial t}\right) = \frac{i}{c}\Gamma_t M_x;$$

hence, with $cJ = i\Gamma_t$, the second of equations (6) is satisfied. We may now conclude that θ satisfies the partial differential equations (5).

I now proceed to the determination of α and β, two independent solutions of (5).

Let θ be a homogeneous function of M_x, M_y, M_z of degree zero; this quantity may be a constant, but in any case it satisfies the partial differential equations (5), viz.

$$\frac{\partial\theta}{\partial y}M_z - \frac{\partial\theta}{\partial z}M_y + \frac{i}{c}\frac{\partial\theta}{\partial t}M_x = 0, \quad \frac{\partial\theta}{\partial x}M_y - \frac{\partial\theta}{\partial y}M_x + \frac{i}{c}\frac{\partial\theta}{\partial t}M_z = 0,$$

$$\frac{\partial\theta}{\partial z}M_x - \frac{\partial\theta}{\partial x}M_z + \frac{i}{c}\frac{\partial\theta}{\partial t}M_y = 0, \quad \frac{\partial\theta}{\partial x}M_x + \frac{\partial\theta}{\partial y}M_y + \frac{\partial\theta}{\partial z}M_z = 0.$$

Now let $l(\theta)$, $m(\theta)$, $n(\theta)$ be three functions of θ which are to one another in the same ratios as M_x, M_y, M_z, and let $p(\theta)$, $q(\theta)$, $r(\theta)$ be three other functions of θ which satisfy the equations

$$pl + qm + rn = 0, \quad p^2 + q^2 + r^2 = 1, \quad qn - rm = il, \quad \text{etc.},$$

then, if α and β are defined by the equations

$$\alpha = xl(\theta) + ym(\theta) + zn(\theta), \quad \beta = xp(\theta) + yq(\theta) + zr(\theta) - ct,$$

it is easy to see that α and β satisfy the same partial differential equations as θ. These quantities, moreover, are not constants, and are generally independent of one another. The analysis then proceeds as follows:—

It follows from equations (2) and (5) that the vector

$$\mathbf{m} = \mathbf{M}f(\alpha,\ \beta)$$

satisfies the partial differential equations

$$c\,\mathrm{rot}\,\mathbf{m} + i\frac{\partial \mathbf{m}}{\partial t} = 0, \quad \mathrm{div}\,\mathbf{m} = 0 \ldots\ldots\ldots\ldots (7),$$

and can be expressed in the forms

$$\mathbf{m} = [\nabla\alpha,\ \nabla\beta]F(\alpha,\ \beta) \equiv \frac{i}{c}\left\{\frac{\partial\beta}{\partial t}\nabla\alpha - \frac{\partial\alpha}{\partial t}\nabla\beta\right\}F(\alpha,\ \beta) \ldots (8).$$

The relations between α and β which are supplied by these equations enable us to determine α and β. The analysis has been given elsewhere,* the result is that α and β should be defined by a pair of equations of type

$$z-ct=\phi(\alpha,\beta)+(x+iy)\theta(\alpha,\beta),\quad z+ct=\psi(\alpha,\beta)-\frac{x-iy}{\theta(\alpha,\beta)}\ldots(9),$$

where $\theta(\alpha,\beta)$, $\phi(\alpha,\beta)$ and $\psi(\alpha,\beta)$ are arbitrary functions of α and β.

2. It should be mentioned that $M^2=0$ is the necessary and sufficient condition that a solution of equations (2) should possess the property that the equations (7) can be satisfied by a function of type $\mathbf{m}=\theta\mathbf{M}$, where θ is a function of x, y, z, t and not a constant. Equations (5) in fact give zero values for the derivates of θ when the condition $M^2=0$ is not satisfied.

To understand the physical meaning of this result let us put $\theta=a+ib$, $\mathbf{m}=\mathbf{h}+i\mathbf{e}$, then the equation $\mathbf{m}=\theta\mathbf{M}$ gives

$$\mathbf{h}=a\mathbf{H}-b\mathbf{E},\quad \mathbf{e}=b\mathbf{H}+a\mathbf{E},$$

whence

$$\left.\begin{aligned}
[\mathbf{eh}]&=(a^2+b^2)\,[\mathbf{EH}]\\
e^2+h^2&=(a^2+b^2)\,(E^2+H^2)\\
e_xe_y+h_xh_y&=(a^2+b^2)\,(E_xE_y+H_xH_y)\\
(e_x^2-e_y^2-e_z^2)+(h_x^2-h_y^2-h_z^2)&=(a^2+b^2)(E_x^2-E_y^2-E_z^2+H_x^2-H_y^2-H_z^2)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\}(10)$$

and so on. It follows from these results that the component stresses, the volume density of energy, and the components of the flow of energy in the field $(\mathbf{e}, \mathbf{h})$ are simple multiples of the corresponding quantities in the field $(\mathbf{E}, \mathbf{H})$, while the velocity at which the energy flows is the same in both cases.† It appears then that if we know the ratios of the stresses and the quantities relating to the energy at each point of space, the field is uniquely determined‡ except when the conditions $E^2-H^2=0$ and $(\mathbf{EH})=0$ are satisfied. In a field of this exceptional nature, energy flows with the velocity of light; and since the field is only one of a class of radiant fields specified by vectors $\mathbf{m}$ of type $\mathbf{M}f(\alpha,\beta)$, where f is an arbitrary

* *American Journal of Mathematics*, April (1915).

† This velocity $\mathbf{V}$ is defined by the equation $\frac{1}{2}(E^2+H^2)\,\mathbf{V}=[\mathbf{EH}]$.

‡ Except as regards a constant complex factor of type θ in the expression for $\mathbf{M}$.

function, it appears that the field can be regarded as built up of individual elements which are practically independent of one another. This leads us to a corpuscular theory for the simple radiant fields of this type, and points to the conclusion that Newton was right in regarding a corpuscular theory as more fundamental than a wave theory of light. This does not mean, however, that a single simple radiant field corresponds to light as we know it. Mathematical analysis indicates that two superposed radiant fields of the simple type are needed to produce the field of an electric charge moving with a velocity less than that of light, and a large number of superposed fields may be required to represent the field of a vibrating atom.

It has already been suggested that the corpuscles or light-particles consist of exceedingly minute particles of electricity which travel in pairs so as to form doublets except when they are within the boundary of an electron or other free charge of electricity. The separation of the constituents of a doublet into free particles of electricity, which travel along straight lines with the velocity of light and very soon secure new partners, gives rise to the phenomenon of a free electric charge or elementary particle of matter.

The breaking up of a doublet is probably due to an encounter with a free particle of electricity which seizes one of the constituents and frees the other. Since the doublets are exceedingly numerous, this new free particle of electricity at once breaks up another doublet, and so on, leading to the phenomenon of a permanent free charge which either remains stationary or moves with a velocity less than the velocity of light.

If we call a point at which doublets continually break up a *source* we naturally wish to have some idea of the number of sources within the boundary of an electron or positive nucleus of an atom. Now, according to the usual theory of electricity, each electron acts on all the electrons and positive charges in the universe; consequently, if this action is due to the breaking up of light-particles fired out from the first electron, the total number of sources within this electron must be of the same order of magnitude as the total number of electrons in the universe, and is thus inconceivably great. On this view a light-particle which has just left an electron must be exceedingly minute, but I do not think this will be a great surprise to physicists; for the idea that the æther is exceedingly fine grained is a familiar one, and scientists who are familiar with the immense distances between stars and the minuteness of an electron, will be quite accustomed to large numbers and extremely minute particles.

Of course it is by no means certain that each electron acts directly on all the others; an assumption to the contrary may be quite compatible with electromagnetic theory, and would not require the existence of particles of such very small size. This aspect of the theory is very interesting, but we shall not discuss it at present.

3. It is well known that equations (2) may be satisfied by writing

$$\mathbf{M} = i \operatorname{rot} \mathbf{A} = \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \nabla \Phi \quad \ldots\ldots\ldots\ldots (11),$$

where the vector $\mathbf{A}$ and the scalar Φ are connected by the equation

$$\operatorname{div} \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0 \quad \ldots\ldots\ldots\ldots\ldots (12),$$

in addition to the one given above. These equations imply that Φ and the three components of $\mathbf{A}$ are wave-functions.

General expressions for $\mathbf{A}$ and Φ may be written down by generalising Whittaker's solution of the wave-equation. Let us write

$$\alpha = x \sin\theta \cos\phi + y \sin\theta \sin\phi + z \cos\theta - ct,$$

$$\beta = x \sin\theta \sin\phi - y \sin\theta \cos\phi + iz - ict \cos\theta,$$

$$\gamma = x \cos\theta - iy - z \sin\theta \cos\phi + ict \sin\theta \sin\phi,$$

$$\delta = x - iy \cos\theta + iz \sin\theta \sin\phi - ct \sin\theta \cos\phi;$$

then, if $\mathbf{L}(\theta, \phi)$ denotes the vector whose components are $\sin\theta\cos\phi$, $\sin\theta\sin\phi$, $\cos\theta$ respectively, and F denotes an arbitrary function of its six arguments, the quantities

$$\mathbf{A} = \int_0^{\pi}\int_0^{2\pi} F(\alpha, \beta, \gamma, \delta, \theta, \phi)\, \mathbf{L}(\theta, \phi)\, d\theta\, d\phi,$$

$$\Phi = \int_0^{\pi}\int_0^{2\pi} F(\alpha, \beta, \gamma, \delta, \theta, \phi)\, d\theta\, d\phi,$$

satisfy the requirements. The integrand itself, of course, provides a particular solution in which the same arbitrary function is a multiplier in Φ and each component of $\mathbf{A}$. This particular solution also possesses the property $A^2 = \Phi^2$, and it is natural to inquire whether this condition is characteristic of a field in which quantities of type $\lambda\mathbf{A}$ and $\lambda\Phi$ possess the same properties as $\mathbf{A}$ and Φ.

If we substitute $\lambda\mathbf{A}$ and $\lambda\Phi$ instead of $\mathbf{A}$ and Φ in the equations

$$i\,\mathrm{rot}\,\mathbf{A}=\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}+\nabla\Phi,\quad \mathrm{div}\,\mathbf{A}+\frac{1}{c}\frac{\partial\Phi}{\partial t}=0\ldots.(13),$$

we find that the equations will be satisfied if λ satisfies the partial differential equations

$$i[\nabla\lambda,\,\mathbf{A}]=\frac{1}{c}\,\mathbf{A}\frac{\partial\lambda}{\partial t}+\Phi\nabla\lambda,\quad (\mathbf{A},\,\nabla\lambda)+\frac{1}{c}\,\Phi\frac{\partial\lambda}{\partial x}=0\ldots(14).$$

These equations give zero values for the derivates of λ except when $A^2=\Phi^2$; hence the above surmise is correct.

The general solution of equations (13), when $\mathbf{A}$ and Φ are connected by the relation $A^2=\Phi^2$, has not yet been found, but an interesting class of solutions may be specified as follows:

Let α and β be functions of x, y, z, t defined by equations (9); then, if $\theta(\alpha,\,\beta)$ is the function occurring in these equations, it is easy to verify that the expressions

$$\mathbf{A}=f(\alpha,\,\beta)\,\nabla\theta,\quad c\Phi=-f(\alpha,\,\beta)\frac{\partial\theta}{\partial t}\ldots\ldots\ldots(15)$$

satisfy all the conditions. The analytical verification is long but straightforward, and so need not be given here. It should be noticed that θ, and in fact an arbitrary function of θ, is a solution of the wave-equation. This result is intimately connected with some theorems due to Forsyth and Whittaker,* it should be noticed in particular that when

$$\phi(\alpha,\,\beta)=\zeta(\alpha)-c\tau(\alpha)-\theta[\xi(\alpha)+i\eta(\alpha)],$$

$$\psi(\alpha,\,\beta)=\zeta(\alpha)+c\tau(\alpha)+(1/\theta)[\xi(\alpha)-i\eta(\alpha)],$$

α is defined by an equation of type

$$[x-\xi(\alpha)]^2+[y-\eta(\alpha)]^2+[z-\zeta(\alpha)]^2=c^2[t-\tau(\alpha)]^2\ldots(16),$$

and θ by the equation

$$\theta=\frac{z-\zeta(\alpha)-c[t-\tau(\alpha)]}{x-\xi(\alpha)-i(y-\eta(\alpha))}\ldots\ldots\ldots\ldots(17).$$

The solution embodied in (15) is closely connected with a solution which has already been mentioned.† If $\mathbf{m}$ denotes a vector given by expressions of type (8), where α and β are

* See the author's *Electrical and Optical Wave Motion*, pp. 119, 136.

† *Ibid.*, p. 125, foot-note.

defined by means of (9), we may generalise this solution by writing

$$\mathbf{A} = i[\mathbf{Pm}] - Q\mathbf{m}, \quad \Phi = (\mathbf{Pm}) \ldots\ldots\ldots (18),$$

where the vector $\mathbf{P}$ and the scalar Q are arbitrary functions of α and β. It is easy to verify that these expressions satisfy the equations (13) and are connected by the relation $A^2 = \Phi^2$.

4. Turning now to the theory of a moving point charge of electricity we shall endeavour to give an alternative proof of a geometrical theorem which has already been proved by another method elsewhere. The theorem may be stated as follows: "Light-particles move to and from a moving point S in such a way that if S_1, S_2, S_3 are any three consecutive positions of the moving point, the light-particle which arrives at S_3 encounters on its journey the light-particles which started from S_1 and S_2. If S is regarded as a moving point charge of electricity the light-particles which leave S lie at any instant on a line of electric force of the field of the point charge, and the light-particles which arrive at S would, if they continued along their rectilinear paths, also lie at each instant on a line of electric force."

Let the coordinates of the point S at time τ be $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$, and let us use dashes to denote differentiations with respect to τ. Let $l(\tau)$, $m(\tau)$, $n(\tau)$ denote the direction cosines of the line of motion of the light-particle which leaves S at time τ, and let $L(\tau)$, $M(\tau)$, $N(\tau)$ denote the direction cosines of the line of motion of the light-particle which arrives at S at time τ. Let τ denote the time when S is in the position S_2, $\tau - \epsilon$, $\tau + \epsilon$ the times when it is in the positions S_1 and S_3 respectively. Let the light-particle which leaves S_1 collide with the light-particles which arrive at S_2 and S_3, at times $\tau - \epsilon + \theta$ and $\tau - \epsilon + \chi$ respectively; also let the light-particle which leaves S_3 collide with the light-particle, which arrives at S_2, at time $\tau + \phi$. We then have sets of equations of the following types in which the Taylor expansions are used:

$$\xi + \epsilon\xi' + \tfrac{1}{2}\epsilon^2\xi'' + \ldots = \xi + cl\phi + c(L + \epsilon L')(\epsilon - \phi),$$

$$= \xi - \epsilon\xi' + \tfrac{1}{2}\epsilon^2\xi'' + \ldots + c(l - \epsilon l')\chi + c(L + \epsilon L')(2\epsilon - \chi),$$

$$\xi - \epsilon\xi' + \tfrac{1}{2}\epsilon^2\xi'' + cL(\epsilon - \theta) + c(l - \epsilon l')\theta = \xi.$$

Neglecting terms of order higher than the second, we deduce that

$$\left.\begin{aligned} \epsilon\xi' + \tfrac{1}{2}\epsilon^2\xi'' &= cl\phi + c(L+\epsilon L')(\epsilon-\phi) \\ 2\epsilon\xi' &= c(l-\epsilon l')\chi + c(L+\epsilon L')(2\epsilon-\chi) \\ \epsilon\xi' - \tfrac{1}{2}\epsilon^2\xi'' &= cL(\epsilon-\theta) + c\theta(l-\epsilon l') \end{aligned}\right\} \ldots(19),$$

whence

$$\epsilon\xi'[\chi(\epsilon-\phi)+\theta(\chi-2\phi)] + \tfrac{1}{2}\epsilon^2\xi''[\chi(\phi-\theta)+\epsilon(2\theta-\chi)] = cL\chi(\epsilon-\theta)(\epsilon-\phi) + cl\theta\phi(2\epsilon-\chi)$$

$$c\epsilon\chi(\epsilon-\phi)\,l' = \epsilon(\xi'-cl)(\chi-2\phi) + \tfrac{1}{2}\epsilon^2\xi''(2\epsilon-\chi).$$

Now $2\theta-\chi$ and $\theta-\phi$ are small quantities which are negligible in comparison with ϵ, we may therefore put them equal to zero in the first of the above equations and also in the second equation when we wish to determine the ratios of the coefficients of l' and ξ''. The first equation now becomes

$$\epsilon\xi' - c\phi l = cL(\epsilon-\phi) \quad \ldots\ldots\ldots\ldots\ldots\ldots(20),$$

and the second may be written in the form

$$cl' = A(\xi'-cl) + \xi''\frac{\epsilon}{2\phi} \quad \ldots\ldots\ldots\ldots\ldots\ldots(21),$$

where A denotes the quantity $\dfrac{\chi-2\phi}{\chi(\epsilon-\phi)}$.

Now the three equations of type (20) give

$$\epsilon^2(\xi'^2+\eta'^2+\zeta'^2) - 2\epsilon c\phi(l\xi'+m\eta'+n\zeta') + c^2\phi^2 = c^2(\epsilon^2-2\epsilon\phi+\phi^2)$$

or $$2\epsilon c\phi(c-l\xi'-m\eta'-n\zeta') = \epsilon^2(c^2-\xi'^2-\eta'^2-\zeta'^2).$$

The ratio of ϵ to $2c\phi$ is now determined; to determine A we multiply the three equations of type (21) by l, m and n respectively and add, this gives

$$A(c-l\xi'-m\eta'-n\zeta') = \frac{\epsilon}{2\phi}(l\xi''+m\eta''+n\zeta'').$$

Hence finally we obtain three equations of type

$$(c^2-\xi'^2-\eta'^2-\zeta'^2)\,l' = (\xi'-cl)(l\xi''+m\eta''+n\zeta'') + \xi''(c-l\xi'-m\eta'-n\zeta') \ldots(22).$$

These, however, are the equations for the variation of the direction of projection of light-particles which form a line of electric force. This proves the first part of the theorem; to prove the second part we notice that equations (19) also give

$$c\epsilon\theta\,(2\epsilon-\chi)\,L'=\epsilon\,(2\theta-\chi)(\xi'-cL)+\tfrac{1}{2}\chi\epsilon^2\xi''.$$

Putting $2\theta=\chi$ and $\theta=\phi$ to determine the ratio of the coefficients of L' and ξ'', we find that

$$\frac{\chi\epsilon^2}{2c\epsilon\theta\,(2\epsilon-\chi)}=\frac{\epsilon}{2c\,(\epsilon-\phi)},$$

and the three equations of type (20) give

$$2c\,(\epsilon-\phi)(c-L\xi'-M\eta'-N\zeta')=\epsilon\,(c^2-\xi'^2-\eta'^2-\zeta'^2);$$

hence it follows that L, M and N satisfy differential equations of the same type as l, m, n. This proves the second part of the theorem, which may also be deduced with the aid of a theorem proved by Murnaghan.*

If the point charge S oscillates, the direction of projection of light-particles belonging to a given line of force also oscillates, and the line assumes a wavy form. In the case when the point charge performs a simple harmonic motion

$$\xi=a\cos(\omega\tau+\alpha),\quad \eta=0,\quad \zeta=0,$$

a wave on the line of force increases in amplitude as it travels out at a rate which is nearly equal to $a\omega$.† The intensity of the light is proportional to the square of the maximum acceleration $a\omega^2$ multiplied by the period of vibration $2\pi/\omega$. Now if ω is very large it is possible for $a^2\omega^2$ to be of appreciable magnitude while $a\omega$ is negligibly small. Thus in the case of light of high frequency the amplitude of the wave which travels along a line of force increases at an exceedingly slow rate. This result is of some interest in connection with Sir Joseph Thomson's idea that ultra-violet light and Röntgen rays consist of pulses in which the energy is not distributed uniformly over the wave-front, but is concentrated in certain regions just as it would be if the radiation consisted of vibrations or kinks travelling along particular lines of force.‡ The advantages and disadvantages of a theory of radiation based on an idea of this kind have been discussed on many occasions;§ it will be sufficient here to remark that many of the objections which have been raised lose their force when we adopt Heavi-

* *John Hopkins Circular* (1915); *Amer. Journ. of Math.*, April (1917).

† See the figure on p. 132 of the author's *Electrical and Optical Wave Motion*, and the explanation on p. 131. The curves shown are particular lines of force, as Murnaghan has shown, *loc. cit.*

‡ *Proc. Camb. Phil. Soc.*, October (1907); *Electricity and Matter* (1903), p. 63.

§ See, for instance, N. R. Campbell, *Modern Electrical Theory*, 2nd Ed. (1913) chapters 10 and 11. J. H. Jeans, "Report on Radiation and the Quantum Theory," *Proc. Phys. Soc. London* (1914).

side's suggestion * and develop a corpuscular theory of radiation with the aid of Maxwell's equations. The solution of these equations that seems most appropriate for a theory of this type has already been discussed in these pages.† It corresponds to a type of point charge which does not differ from an ordinary point charge, as far as its field is concerned, when it is moving with uniform velocity along a straight line. As soon as its velocity varies a wave starts out in which energy is concentrated in such a way that a curve, which may be a particular line of force in the field of an ordinary point charge moving in the same way, becomes a line along part of which the field vectors are infinite. In addition to the concentrated disturbance running along this singular line of force, there is a disturbance which spreads out in all directions so that the radiation is of a dual nature. The difficulty of explaining the phenomena of diffraction and interference with a corpuscular or "kink" theory of radiation ‡ is not insuperable when the present theory is adopted, for these phenomena may be produced by the waves which spread out in all directions. § Another objection which has been raised against Thomson's theory is that the kink widens out considerably and its energy becomes distributed along a lengthening line, and so is not sufficiently concentrated to produce the intense localised effects which must take place in ionisation and in the ejection of electrons with high velocities. This objection, however, loses its force when we remember that these phenomena are produced chiefly by light of high frequency, and, as we have just seen, the rate at which a kink widens out is then negligibly small.

5. In some of Sir Joseph Thomson's applications of the theory of Faraday tubes to chemistry ‖ it is assumed that a unit positive charge and a unit negative charge are joined by only one singular Faraday tube. When each of the charges

* *Electromagnetic Theory*, vol. iii, p. 477; *The Electrician*, Nov. 29 (1901). See also T. Levi Civita, *Comptes Rendus*, t. 145 (1907), p. 417; *Il nuovo cimento*, t. 18 (1909), p. 163; *Rend. Lincei*, vol. xviii. (1909), p. 83.

† *Messenger of Mathematics*, May (1915).

‡ The difficulties are discussed by H. A. Lorentz, *Phys. Zeitschr*, Bd. 11 (1910), p. 349; J. J. Thomson, *Proc. Camb. Phil. Soc.*, vol. xvi. (1912), p. 643; G. I. Taylor, *Ibid.*, vol. xv. (1909), p. 114; Kerschbaum, *Phil. Mag*, vol. xxvii. (1914), p. 397.

§ In the discussion of his photo-electric experiments, E. Marx, *Ann. d. Phys.*, Bd. 61 (1913), p. 167, thinks of waves with points of concentration, and finds that the amount of energy localised in these points need not be more than a small fraction of the energy contained in the wave.

‖ See, for instance, "The Corpuscular Theory of Matter" (1907), chap. vi.; *Phil. Mag.* [6], vol. xxvii. (1914), p. 757.

is moving so that radiation takes place, it must presumably be supposed that energy flows along the tube in both directions.* This raises the question as to whether it is possible for a single moving curve to be the locus at each instant of two systems of light-particles fired out from two moving charges A and B respectively.

In order that this may be the case it is evident in the first place that the straight lines described by one set of light-particles must each intersect all the straight lines described by the other set of light-particles, consequently the two sets of lines either lie in a plane or are generators of a hyperboloid.

Now consider a quadrilateral $PQRS$ formed by two pairs of rays belonging to different systems. If t_P, t_Q, t_R, t_S denote the times at which light-particles are at the corners of the quadrilateral, we have

$$c(t_Q - t_P) = PQ,\ c(t_Q - t_R) = RQ,\ c(t_P - t_S) = SP,\ c(t_R - t_S) = SR,$$

where c is the velocity of light, consequently

$$PQ + SP = c(t_Q - t_S) = RQ + SR.$$

Now this is a property of lines drawn on a hyperboloid of revolution and of lines in a plane which either touch a circle or form a parallelogram. Hence the paths of the two moving charges A and B must either lie in one plane or on a hyperboloid of revolution.

In the former case, if any four rays touch a circle C, it follows that all the rays must touch the same circle. Now consider two points L and M which are reached simultaneously by light-particles. If rays through these points meet in K and N we must have $LN = MN$ and $LK = MK$. Hence L and M are equidistant from the centre of the circle C; it follows then that all the points which are reached simultaneously lie on a circle concentric with C. In particular, A and B must lie on such a circle at each instant.

To find the way in which the radius R of this circle varies with the time, we consider the motion of a light-particle which is on the circle C at the point N at time t_n; then, if it is at a point P on the tangent at N at time t, we have

$$R^2 = a^2 + c^2(t - t_n)^2,$$

where a is the radius of the circle C. It appears from this equation that R is infinite when $t = -\infty$; it decreases to the

* This idea is used by Sir Joseph Larmor in his mechanical model of the pressure of radiation, *Camb. Math. Congress*, vol. i., p. 202; see also T. Levi Civita, *ibid.*, p. 207.

minimum value a and then increases again indefinitely; thus the curvature of the line of force increases to a maximum and then decreases.

If any two pairs of rays belonging to the two systems form a parallelogram, all the rays of one system must be in one direction, and all the rays of the other system in another. In this case, if L and M are reached simultaneously, $LKMN$ is a rhombus, and so LM bisects the angle between the two fixed directions. This means that all the points which are reached simultaneously lie on a straight line, which is always in a fixed direction. The rate at which this line moves is easily found by considering the motion of one light-particle. The velocity is evidently $c\sin\theta$, where θ is the angle between the line and the path of the light-particle.

It is clear from the above that if two moving points A and B are joined by only one singular Faraday tube along which disturbances can flow in both directions, the motions of the two points must be of a very restricted nature.

If the idea of singular Faraday tubes is to be used to elucidate the properties of chemical valence, it may be necessary to assume that a unit charge carries two singular Faraday tubes—one along which energy is radiated away from the charge, and another along which energy is radiated to the charge. It may be that in a simple type of chemical combination two electrons have their singular Faraday tubes attached to the positive nuclei of both of the combined atoms, and that energy flows in a cycle along the four Faraday tubes. The idea that two electrons take part in a simple chemical combination is one which has been familiar to chemists for some time.

STIRLING'S ASYMPTOTIC FORMULA FOR $\Gamma(x+1)$

By *A. E. Jolliffe, M.A.*

I PROPOSE to obtain by elementary methods, without the use of the calculus, this formula in the form

$$\log\Gamma(x+1)=(x+\tfrac{1}{2})\log x-x+\tfrac{1}{2}\log(2\pi)+\frac{1}{12x}-\frac{\theta}{360x^3},\quad 0<\theta<1,$$

for all positive values of x.

I take $\Gamma(x+1)$ to be defined as

$$\lim_{n\to\infty} n^x(n!)/\{(x+1)(x+2)\ldots(x+n)\}.$$

The logarithms are to be base e.

I shall first prove the formula for the case of x a positive integer n, so that $\Gamma(n+1)=n!$ and thence deduce the general case. The method adopted for the first case is different to that given in Chrystal's *Algebra*, and obtains the result, which is a much closer approximation than that given by Chrystal, with very much less trouble. An extension to the general case by elementary methods I have not seen before.

1. *The case of x a positive integer n.*

We have $(n+\frac{1}{2})\log n-(n-\frac{1}{2})\log(n-1)$

$$=\log n+(n-\tfrac{1}{2})\log\frac{n}{n-1}=\log n+\frac{1}{2u}\log\frac{1+1/u}{1-1/u},$$

where $u\equiv 2n-1$,

$$=\log n+1+\frac{1}{3u^3}+\frac{1}{5u^4}+\ldots .$$

If we write down the results obtained by changing n into $n-1$, $n-2$, ..., 2 and add them together, we obtain

$$(n+\tfrac{1}{2})\log n=\log(n!)+n-1+\tfrac{1}{3}\sum_2^n\frac{1}{(2n-1)^2}+\tfrac{1}{5}\sum_2^n\frac{1}{(2n-1)^4}+\ldots$$

Now the double series of *positive* terms

$$\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots,$$

$$+\frac{1}{3^4}+\frac{1}{5^4}+\ldots,$$

$$+\frac{1}{3^6}+\ldots,$$

$$+\ldots$$

is convergent, for its sum by columns is

$$\frac{1}{2.4}+\frac{1}{4.6}+\frac{1}{6.8}+\ldots, \text{ which is equal to } \tfrac{1}{4}.$$

Hence the double series

$$\frac{1}{3}\cdot\frac{1}{3^2}+\frac{1}{3}\cdot\frac{1}{5^2}+\frac{1}{3}\cdot\frac{1}{7^2}+\ldots,$$

$$+\frac{1}{5}\cdot\frac{1}{3^4}+\frac{1}{5}\cdot\frac{1}{5^4}+\ldots,$$

$$+\frac{1}{7}\cdot\frac{1}{3^6}+\ldots,$$

$$+\ldots$$

is convergent, and, if its sum is $1-A$, we have

$$\log(n!)=(n+\tfrac{1}{2})\log n-n+A+\tfrac{1}{3}\sum_n^{\infty}\frac{1}{(2n+1)^2}+\tfrac{1}{5}\sum_n^{\infty}\frac{1}{(2n+1)^4}+\ldots$$

The terms on the right-hand side after the first three may be written as

$$\tfrac{1}{3}\sum_n^{\infty}\frac{1}{(y+1)}+\tfrac{1}{5}\sum_n^{\infty}\frac{1}{(y+1)^2}+\tfrac{1}{7}\sum_n^{\infty}\frac{1}{(y+1)^3}+\ldots,$$

where y denotes $4n^2+4n$. Now

$$\tfrac{1}{3}\frac{1}{y+1}+\tfrac{1}{5}\frac{1}{(y+1)^2}+\tfrac{1}{7}\frac{1}{(y+1)^3}+\ldots$$

is less than

$$\tfrac{1}{3}\frac{1}{y+1}\left(1-\frac{1}{y+1}\right)^{-1},\text{ that is }\frac{1}{3y}.$$

Also $\sum_n^{\infty}\frac{1}{y}=\frac{1}{4n}$, so that the sum of these terms is less than $\frac{1}{12n}$.

Again
$$\frac{1}{y+1}=\frac{1}{y}-\frac{1}{y^2}+\frac{1}{y^2(y+1)},$$

and
$$\frac{1}{(y+1)^2}=\frac{1}{y^2}-\frac{2y+1}{y^2(y+1)^2}.$$

Also $\frac{1}{n}-\frac{1}{n+1}=\frac{4}{y}$, therefore $\frac{1}{n}=4\Sigma\frac{1}{y}$;

and

$$\frac{1}{n^3}-\frac{1}{(n+1)^3}=\frac{48}{y^2}+\frac{64}{y^3},\text{ therefore }\frac{1}{n^3}=48\Sigma\frac{1}{y^2}+64\Sigma\frac{1}{y^3}.$$

Therefore $\frac{1}{12n}-\frac{1}{360n^3}=\frac{1}{3}\Sigma\frac{1}{y}-\frac{2}{15}\Sigma\frac{1}{y^2}-\frac{8}{45}\Sigma\frac{1}{y^3}$,

so that
$$\tfrac{1}{3}\Sigma\frac{1}{y+1}+\tfrac{1}{5}\Sigma\frac{1}{(y+1)^2}+\ldots$$

is greater than

$$\frac{1}{12n} - \frac{1}{360n^3} + \tfrac{8}{45}\Sigma\frac{1}{y^3} + \frac{1}{3y^2(y+1)} - \frac{2y+1}{5y^2(y+1)^2},$$

which is greater than

$$\frac{1}{12n} - \frac{1}{360n^3} + (\tfrac{1}{3} - \tfrac{2}{5} + \tfrac{8}{45})\,\Sigma\frac{1}{y^2(y+1)},$$

which is greater than $\quad \dfrac{1}{12n} - \dfrac{1}{360n^3}$.

Hence we have

$$\log(n!) = (n+\tfrac{1}{2})\log n - n + A + \frac{1}{12n} - \frac{\theta}{360n^3}, \quad 0<\theta<1.$$

The determination of A can stand over for the present. With a little more trouble we could replace the term

$$\frac{\theta}{360n^3} \text{ by } \frac{1}{360n^3} - \frac{\theta}{1260n^5}, \quad 0<\theta<1,$$

but it is hardly worth while to give the details.

2. *The general case.*

We have

$$\log\Gamma(x+1) = \lim_{n\to\infty}\{x\log n + \log(n!) - \sum_1^n \log(x+n)\},$$

Now

$$\begin{aligned}&(x+n+\tfrac{1}{2})\log(x+n) - (x+n-\tfrac{1}{2})\log(x+n-1)\\ &= \log(x+n) + (x+n-\tfrac{1}{2})\log\frac{x+n+\frac{1}{2}}{x+n-\frac{1}{2}}\\ &= \log(x+n) + \tfrac{1}{2}u\log\frac{u+1}{u-1}, \text{ where } u\equiv 2x+2n-1.\end{aligned}$$

Hence, proceeding as before, we obtain

$$\begin{aligned}&(x+n+\tfrac{1}{2})\log(x+n) - (x+\tfrac{1}{2})\log x\\ &\qquad = \sum_1^n \log(x+r) + n + \tfrac{1}{3}\sum_1^n\frac{1}{y+1} + \tfrac{1}{5}\sum_1^n\frac{1}{(y+1)^2} + \cdots,\end{aligned}$$

where y_r denotes $4(x+r)(x+r-1)$.

Hence $\quad \log\Gamma(x+1) = (x+\tfrac{1}{2})\log x + \dfrac{1}{12x} - \dfrac{\theta}{360x^3}$

$$+ \lim_{n\to\infty}\{x\log n + \log(n!) - (x+n+\tfrac{1}{2})\log(x+n) + n\}.$$

The expression inside the brackets is

$$x\log n+(n+\tfrac{1}{2})\log n+A+\frac{1}{12n}-\frac{\theta'}{360n^3}-(x+n+\tfrac{1}{2})\log(x+n)$$

or $$A+\frac{1}{12n}-\frac{\theta'}{360n^3}-n\log\left(1+\frac{x}{n}\right)-(x+\tfrac{1}{2})\log\left(1+\frac{x}{n}\right),$$

the limit of which, as n tends to infinity, is $A-x$. Hence

$$\log\Gamma(x+1)=(x+\tfrac{1}{2})\log x-x+A+\frac{1}{12x}-\frac{\theta}{360x^3},\quad 0<\theta<1.$$

The developments in series only require that $2x+1$ should be greater than unity, so that the result is valid for all positive values of x.

Since the value of A is $\frac{1}{2}\log(2\pi)$, its determination by very elementary methods is quite out of the question. I have no new method to suggest, and therefore can only refer to the two usual methods, which are as follows. We have

$$\lim_{n\to\infty}\frac{2^{2n}(n!)^2}{(2n)!}n^{-\frac{1}{2}}=\lim_{n\to\infty}\frac{2.4.6\ldots 2n}{1.3.5\ldots 2n-1}n^{-\frac{1}{2}}=\pi^{\frac{1}{2}}$$

(Wallis' Identity).

Or we can prove from the definition of $\Gamma(x+1)$ that

(i) $\Gamma(x+\frac{1}{2})\Gamma(x+1)=2^{-2x}\Gamma(\frac{1}{2})\Gamma(2x+1)$,

(ii) $\Gamma(p)\Gamma(1-p)=\pi\operatorname{cosec}\pi p$, whence $\Gamma(\frac{1}{2})=\pi^{\frac{1}{2}}$.

In either case, by taking logarithms of both sides and using the expression found above for $\Gamma(x+1)$, we find A to be $\frac{1}{2}\log 2\pi$.

A very short method of finding A is given by Dr. Bromwich*. This consists in writing the double-series

$$\tfrac{1}{3}\Sigma\frac{1}{3^2}+\tfrac{1}{5}\Sigma\frac{1}{5^2}+\ldots\text{as}\int_0^1\sum_1^\infty\frac{t^2}{(2n+1)^2-t^2}\,dt,$$

that is

$$\int_0^1\left(\frac{\pi t}{4}\tan\frac{\pi t}{2}-\frac{t^2}{1-t^2}\right)dt,$$

which is equal to $1-\frac{1}{2}\log(2\pi)$. But this method depends upon the calculus and, as it requires a knowledge of the fact that

$$\int_0^{\frac{1}{2}\pi}\log\cos\theta\,d\theta=\frac{\pi}{2}\log\tfrac{1}{2},$$

cannot be regarded as elementary.

* *Messenger of Mathematics*, vol. xxxvi., p. 83.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

XLIX.

On Mellin's inversion formula.

1. The relationship between the two functions f and ϕ, expressed by the reciprocal equations

$$f(s) = \int_0^\infty \phi(x) x^{s-1} dx \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and

$$\phi(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} f(s) x^{-s} ds \ldots\ldots\ldots\ldots (2),$$

was (so far as I am aware) first observed by Riemann in his famous memoir on the distribution of primes.* Riemann deduces (2) from (1) by a purely formal process based on Fourier's integral theorem; and the first accurate discussion of the question was that of Mellin,† with whose name the formulæ are generally associated.

Mellin bases his discussion on a somewhat restrictive hypothesis as to the nature of $\phi(x)$, viz. that it is an analytic function regular (except at the origin and infinity) throughout an angular region including the positive real axis; and the formulæ are so interesting that a more general discussion seems called for. Such a discussion will also form a natural sequel to some recent notes of this series, which deal with problems of a very similar character.‡

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* B. Riemann, 'Ueber die Anzahl der Primzahlen unter einer gegebener Grösse', *Werke*, (ed. 1876), pp. 136–144 (p. 140). Riemann's formulæ are not quite the same as (1) and (2), but embody the same idea.

† Hj. Mellin, 'Ueber die fundamentale Wichtigkeit des Satzes von Cauchy für die Theorien der Gamma- und der hypergeometrischen Funktionen', *Acta Societatis Fennicae*, vol. xxi., 1896, no. 1, pp. 1–115, §§ 14–29; 'Abriss einer einheitlichen Theorie der Gamma- und der hypergeometrischen Funktionen', *Math. Annalen*, vol. lxviii., 1910, pp. 305–337, § 8.

‡ Note XLII, 'On Weierstrass's singular integral'; Notes XLVI and XLVII, 'On Stieltjes' *problème des moments*': *Messenger of Mathematics*, vol. xlvi., 1917, pp. 43–48, 175–182; vol. xlvii., 1918, pp. 81–88.

2. In discussing the integral (2) I shall take as fundamental not the ordinary idea of convergence of an infinite integral but that of *summability*. I shall say that the integral

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} f(s)x^{-s}ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(a+it)x^{-a-it}dt$$

is *summable* $(C, 1)$ if

$$\frac{1}{2\pi T}\int_{-T}^{T}(T-|t|)f(a+it)x^{-a-it}dt$$

tends to a limit when $T\to\infty$ *. It is easily verified that this definition satisfies the 'condition of consistency', *i.e.* that a convergent integral is summable to a sum equal to its value.

THEOREM A. *Suppose that* $\phi(x)$ *is integrable (in the sense of Lebesgue*†*) in every interval* $0<\delta\leq x\leq\Delta$; *and that*

$$\int_0 x^{\alpha-1}\phi(x)dx, \qquad \int x^{\beta-1}\phi(x)dx,$$

where $\alpha<\beta$, *are convergent*‡, *so that*

$$\int_0^{\infty} x^{s-1}\phi(x)dx,$$

where $s=\sigma+it$, *is uniformly convergent throughout any strip* $\alpha<\alpha'\leq\sigma\leq\beta'<\beta$, *and represents an analytic function* $f(s)$ *regular in the strip. Then the integral*

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} f(s)x^{-s}ds \quad (\alpha<a<\beta)$$

is summable $(C, 1)$, *to sum* $\phi(x)$, *for almost all positive values of* x. *In particular this is true at all points of continuity of* $\phi(x)$: *more generally, the integrable is summable to sum*

$$\tfrac{1}{2}\{\phi(x-0)+\phi(x+0)\}$$

whenever this expression has a meaning.

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* On p. 86 of Note XLVII, in Theorem F, $4\pi iT$ should be $2\pi iT$.

† I use 'integrable' rather than 'summable' in order to avoid using one word in two quite different senses. As Prof. Young has justly remarked, Lebesgue's idea has superseded the idea of 'Riemann integrability' so completely that it is quite unnecessary to mark the distinction any longer by the use of different words.

‡ Not necessarily absolutely.

We have

$$\frac{1}{2\pi i T}\int_{a-iT}^{a+iT}(T-|t|)x^{-s}f(s)\,ds$$

$$=\frac{1}{2\pi T}\int_{-T}^{T}(T-|t|)\,dt\int_{0}^{\infty}y^{a+it-1}\phi(xy)\,dy$$

$$=\frac{1}{2\pi T}\int_{0}^{\infty}y^{a-1}\phi(xy)\,dy\int_{-T}^{T}(T-|t|)y^{it}\,dt,$$

the inversion of the order of integration being permissible in virtue of the uniform convergence of the integral with respect to y. But

$$\int_{-T}^{T}(T-|t|)y^{it}\,dt=2\int_{0}^{T}(T-t)\cos(t\log y)\,dt$$

$$=2\frac{1-\cos(T\log y)}{(\log y)^2};$$

and so

$$\frac{1}{2\pi i T}\int_{a-iT}^{a+iT}(T-|t|)x^{-s}f(s)\,ds$$

$$=\frac{1}{2\pi T}\int_{0}^{\infty}y^{a-1}\phi(xy)\left\{\frac{\sin(\frac{1}{2}T\log y)}{\frac{1}{2}\log y}\right\}^2dy.$$

We write the integral on the right-hand side in the form

$$\frac{1}{2\pi T}\left(\int_{0}^{\delta}+\int_{\delta}^{\Delta}+\int_{\Delta}^{\infty}\right)=J_1+J_2+J_3,$$

say; and we begin by considering J_3.

Let $$\int_{y}^{\infty}w^{a-1}\phi(xw)\,dw=\Phi(y),$$

so that $\Phi(y)\to 0$ when $y\to\infty$. Then

$$\pi J_3=-\frac{1}{T}\int_{\Delta}^{\infty}\Phi'(y)\frac{1-\cos(T\log y)}{(\log y)^2}\,dy$$

$$=\frac{\Phi(\Delta)}{T(\log\Delta)^2}\{1-\cos(T\log\Delta)\}+\int_{\Delta}^{\infty}\Phi(y)\frac{\sin(T\log y)}{y(\log y)^2}\,dy$$

$$-\frac{2}{T}\int_{\Delta}^{\infty}\Phi(y)\frac{1-\cos(T\log y)}{y(\log y)^3}\,dy.$$

Of these three terms, the second (which is obviously the most important) is in absolute value less than

$$\mu \int_{\Delta}^{\infty} \frac{dy}{y(\log y)^2} = \frac{\mu}{\log \Delta},$$

where μ is the upper bound of $|\Phi|$ in (Δ, ∞); and this tends to zero when $\Delta \to \infty$, uniformly in T. It is plain that the same is true of the other two terms; and we can therefore choose Δ so that

$$|J_3| < \epsilon$$

for all values of T greater than unity. A similar argument shews that δ may be chosen so that

$$|J_1| < \epsilon$$

for all values of T greater than unity.

On the other hand, if we write $y = e^u$, $\delta = e^w$, $\Delta = e^\Omega$, we obtain

$$J_2 = \frac{1}{2\pi T} \int_{\omega}^{\Omega} e^{au} \phi(xe^u) \left(\frac{\sin \frac{1}{2} Tu}{\frac{1}{2} u} \right)^2 du.$$

But this is an integral of Fejér's type. It tends to $\phi(x)$ almost always, and in particular whenever x is a point of continuity; and to $\frac{1}{2}\{\phi(x-0)+\phi(x+0)\}$ whenever this expression has a meaning. From this, and our previous results about J_1 and J_3, our theorem follows.

3. We can obtain practically the same result by using other definitions of summability, and in particular that employed for the solution of Stieltjes' problem in Notes XLVI and XLVII.

THEOREM B. *The conclusions of Theorem A are also true if the value of the summable integral is defined as the limit of*

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\eta s^2} f(s) x^{-s} ds \quad (\eta > 0)$$

when η tends to zero.

We have

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\eta s^2} f(s) x^{-s} ds = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\eta s^2} ds \int_0^{\infty} \phi(xy) y^{s-1} dy$$

$$= \frac{1}{2\pi i} \int_0^{\infty} \phi(xy) \frac{dy}{y} \int_{a-i\infty}^{a+i\infty} e^{\eta s^2} y^s ds,$$

if the inversion is permissible. Now this would certainly be the case, in virtue of the uniform convergence of the integral with respect to s, if the limits of integration with respect to y were δ and Δ. It is therefore sufficient to show that

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\eta s^2} ds \int_{\Delta}^{\infty} \phi(xy) y^{s-1} dy \ldots\ldots\ldots\ldots (3)$$

is convergent and tends to zero when $\Delta \to \infty$; and that a similar condition is satisfied for the interval $(0, \delta)$.

Choose c so that $a < a + c < \beta$; and write

$$\int_{\nu}^{\infty} \phi(xw) w^{a+c-1} dw = \Phi(y).$$

Then

$$\int_{\Delta}^{\infty} \phi(xy) y^{a+it-1} dy = -\int_{\Delta}^{\infty} \Phi'(y) y^{it-c} dy$$

$$= \Phi(\Delta)\Delta^{it-c} + (it-c)\int_{\Delta}^{\infty} \Phi(y) y^{it-c-1} dy.$$

The first term tends to zero when Δ tends to infinity, uniformly in t; the second is less in absolute value than a constant multiple of

$$\Delta^{-c}|t|.$$

It follows that (3) is convergent, and less in absolute value than a constant multiple of

$$\Delta^{-c}\int_{-\infty}^{\infty} e^{-\eta t^2}|t|\,dt,$$

which tends to zero. A similar argument may be applied to the interval $(0, \delta)$. Thus the inversion of the order of integration is justified. Now

$$\int_{a-i\infty}^{a+i\infty} e^{\eta s^2} y^s ds = i\int_{-\infty}^{\infty} e^{-\eta t^2} y^{it} dt = i\sqrt{\left(\frac{\pi}{\eta}\right)} e^{-(\log y)^2/4\eta};$$

so that

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\eta s^2} f(s) x^{-s} ds = \frac{1}{2\sqrt{(\pi\eta)}}\int_{0}^{\infty} \phi(xy)\, e^{-(\log y)^2/4\eta}\,\frac{dy}{y}.$$

The limit of the right-hand side, when $\eta \to 0$, may be calculated by putting $y = e^u$ and applying the known results concerning Weierstrass's singular integral.* We thus complete the proof of Theorem B.

* See Note XLII.

4. Neither of the theorems establishes the *convergence* of (2), and this is an essentially more difficult question. We have

$$\frac{1}{2\pi i}\int_{a-iT}^{a+iT} x^{-s} f(s)\,ds = \frac{1}{2\pi}\int_{-T}^{T} dt \int_0^\infty y^{a+it-1}\phi(xy)\,dy$$

$$= \frac{1}{\pi}\int_0^\infty y^{a-1}\phi(xy)\,\frac{\sin(T\log y)}{\log y}\,dy.$$

On writing e^u for y, we obtain an integral of Dirichlet's type, and the conditions that it should tend to a limit are of a more special character. In the first place we must suppose that $\phi(x)$ satisfies (as in the theory of the convergence of Fourier series) some condition more stringent than that of (*e.g.*) mere continuity. Secondly we must assume more about the integral (1) than mere convergence.*

Suppose for example that

$$\phi(x) = \sin x,$$

$$f(s) = \int_0^\infty \sin x\, x^{s-1}\,dx = \sin\tfrac{1}{2}s\pi\,\Gamma(s) \quad (-1 < \sigma < 1).$$

Then $\phi(x)$ is continuous and of bounded variation in any finite interval. But

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \sin\tfrac{1}{2}s\pi\,\Gamma(s)\,x^{-s}\,ds$$

is only convergent if $a < \frac{1}{2}$.

It is easily shown that the integral is convergent, when defined as

$$\lim \int_{a-iT}^{a+iT} f(s)\,x^{-s}\,ds,$$

if $\phi(x)$ satisfies any of the ordinary conditions which occur in the theory of convergence of Fourier series and the integral (1) is *absolutely* convergent for $\alpha < \sigma < \beta$. In particular cases the convergence of the integral, in the ordinary sense, can usually be recognised directly.

5. I conclude by remarking that an elegant formula given recently by Mr. Ramanujan† may be deduced without difficulty from (1) and (2).

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* See, *e.g.*, my paper 'Fourier's double integral and the theory of divergent integrals', *Trans. Camb. Phil. Soc.*, vol. xxi, pp. 427–451 (p. 434).

† S. Ramanujan, 'Some definite integrals', *Messenger of Mathematics*, vol. xliv., 1915, pp. 10–18 [formula (2)].

Take $$\phi(x) = x^p(1+x)^{-q} \quad (0<p<q)$$

so that $$f(s) = \frac{\Gamma(s+p)\,\Gamma(q-p-s)}{\Gamma(q)}$$

if $-p<\sigma<q-p$. We obtain

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} y^{-s}\Gamma(s+p)\,\Gamma(q-p-s)\,ds = \Gamma(q)\,y^p(1+y)^{-q}$$

if $-p<a<q-p$.* If in particular we take $q=2p$, $a=0$, we find

$$\int_0^\infty \cos(t\log y)\,|\,\Gamma(p+it)\,|^2\,dt = \pi\Gamma(2p)\,y^p(1+y)^{-2p};$$

or, writing $y=e^{2m}$ and using the multiplication formula for the Gamma-function,

$$\int_0^\infty \cos(2mt)\,|\,\Gamma(p+it)\,|^2\,dt = \tfrac{1}{2}\sqrt{\pi}\,\Gamma(p)\,\Gamma\left(p+\tfrac{1}{2}\right)\operatorname{sech}^p m,$$

which is Mr. Ramanujan's formula.

In order to give an example which is not covered by Mellin's conditions, we may take

$$\phi(x) = (1-x)^{q-1} \quad (0<x<1), \quad \phi(x) = 0 \quad (x>1),$$

where q is positive. We find that

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\Gamma(s)}{\Gamma(s+q)}\,x^{-s}\,ds = \frac{1}{\Gamma(q)}\,(1-x)^{q-1}$$

if $a>0$, $0<x<1$. If $x>1$, the value of the integral is zero.

THEOREMS CONNECTED WITH MINORS OF AN m-BY-m^2 ARRAY.

By *Sir Thomas Muir, LL.D.*

1. In recently dealing with the question of the resolvability of a certain type of compound determinant an identity came to the surface which has led to several results that seem worthy of being put on record. They are somewhat difficult to classify, and consequently the title of the paper is only an approximate guide to their character. They all concern compound determinants.

---

* This is formula (64) of Mellin's paper in the *Math. Annalen* quoted above.

2. The fundamental theorem may be formulated thus: *If the m^2 columns of an m-by-m^2 array be separated into $m+1$ sets, namely, m sets of $m-1$ columns each and a single set of m columns, then the determinant whose $(r, s)^{\text{th}}$ element is the determinant formed from the r^{th} set and the s^{th} column of the last set has the determinant of the last set for a factor, the cofactor being the compound determinant formed from the first m sets as the former compound determinant was formed from all the sets.* For example, when n is 4, and $|p, q, t, u|$ is used to stand for the determinant whose columns are the p^{th}, q^{th}, t^{th}, u^{th} columns of the array

$$\begin{matrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & g_1 & h_1 & i_1 & j_1 & k_1 & l_1 & x_1 & y_1 & z_1 & w_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_4 & b_4 & c_4 & d_4 & e_4 & f_4 & g_4 & h_4 & i_4 & j_4 & k_4 & l_4 & x_4 & y_4 & z_4 & w_4, \end{matrix}$$

we have

$$\begin{vmatrix} |1,2,3,13| & |1,2,3,14| & |1,2,3,15| & |1,2,3,16| \\ |4,5,6,13| & |4,5,6,14| & |4,5,6,15| & |4,5,6,16| \\ |7,8,9,13| & |7,8,9,14| & |7,8,9,15| & |7,8,9,16| \\ |10,11,12,13| & |10,11,12,14| & |10,11,12,15| & |10,11,12,16| \end{vmatrix}$$

$$= |13,14,15,16| \cdot \begin{vmatrix} |1,2,3,10| & |1,2,3,11| & |1,2,3,12| \\ |4,5,6,10| & |4,5,6,11| & |4,5,6,12| \\ |7,8,9,10| & |7,8,9,11| & |7,8,9,12| \end{vmatrix},$$

or, say $\qquad \Delta_{4,4} = |13,14,15,16| \,.\, \Gamma_{3,4}.$

For, by the multiplication-theorem we obtain

$$\begin{vmatrix} |a_1 b_2 c_3| & |a_1 b_2 c_4| & |a_1 b_3 c_4| & |a_2 b_3 c_4| \\ |d_1 e_2 f_3| & |d_1 e_2 f_4| & |d_1 e_3 f_4| & |d_2 e_3 f_4| \\ |g_1 h_2 i_3| & |g_1 h_2 i_4| & |g_1 h_3 i_4| & |g_2 h_3 i_4| \\ |j_1 k_2 l_3| & |j_1 k_2 l_4| & |j_1 k_3 l_4| & |j_2 k_3 l_4| \end{vmatrix} \cdot \begin{vmatrix} x_4 - x_3 & x_2 - x_1 \\ y_4 - y_3 & y_2 - y_1 \\ z_4 - z_3 & z_2 - z_1 \\ w_4 - w_3 & w_2 - w_1 \end{vmatrix}$$

$$= \begin{vmatrix} |a_1 b_2 c_3 x_4| & |a_1 b_2 c_3 y_4| & |a_1 b_2 c_3 z_4| & |a_1 b_2 c_3 w_4| \\ |d_1 e_2 f_3 x_4| & |d_1 e_2 f_3 y_4| & |d_1 e_2 f_3 z_4| & |d_1 e_2 f_3 w_4| \\ |g_1 h_2 i_3 x_4| & |g_1 h_2 i_3 y_4| & |g_1 h_2 i_3 z_4| & |g_1 h_2 i_3 w_4| \\ |j_1 k_2 l_3 x_4| & |j_1 k_2 l_3 y_4| & |j_1 k_2 l_3 z_4| & |j_1 k_2 l_3 w_4| \end{vmatrix}$$

$$= \Delta_{4,4};$$

and as on changing x, y, z into j, k, l this degenerates into

$$\begin{aligned}||a_1 b_2 c_3|\ |d_1 e_2 f_4|\ |g_1 h_3 i_4|\ |j_2 k_3 l_4||\cdot|j_1 k_2 l_3 w_4| \\ = \begin{vmatrix} |a_1 b_2 c_3 j_4| & |a_1 b_2 c_3 k_4| & |a_1 b_2 c_3 l_4| \\ |d_1 e_2 f_3 j_4| & |d_1 e_2 f_3 k_4| & |d_1 e_2 f_3 l_4| \\ |g_1 h_2 i_3 j_4| & |g_1 h_2 i_3 k_4| & |g_1 h_2 i_3 l_4| \end{vmatrix} \cdot |j_1 k_2 l_2 w_4| \\ = \Gamma_{3,4} \cdot |j_1 k_2 l_3 w_4|,\end{aligned}$$

from which the common factor $|j_1 k_2 l_3 w_4|$ can be removed, there results on substitution

$$\Gamma_{3,4} \cdot |x_1 y_2 z_3 w_4| = \Delta_{4,4}$$

as desired.

3. The condensation of

$$||a_1 b_2 c_3| \cdot |d_1 e_2 f_4| \cdot |g_1 h_3 i_4| \cdot |j_2 k_3 l_4||\ \text{into}\ \Gamma_{3,4},$$

that is to say, the condensation of an n-line compound with $(n-1)$-line elements into an $(n-1)$-line compound determinant with n-line elements, can be accomplished in a variety of ways. Thus, by cyclical permutation of the sets

$$abc,\ def,\ ghi,\ jkl,$$

we obtain as equivalents for $\Gamma_{3,4}$

$$- \begin{vmatrix} |d_1 e_2 f_3 a_4| & |d_1 e_2 f_3 b_4| & |d_1 e_2 f_3 c_4| \\ |g_1 h_3 i_3 a_4| & |g_1 h_2 i_3 b_4| & |g_1 h_2 i_3 c_4| \\ |j_1 k_2 l_3 a_4| & |j_1 k_2 l_3 b_4| & |j_1 k_2 l_3 c_4| \end{vmatrix}, \ \&c.,$$

the general theorem being: *If $n(n-1)$ columns of n elements each be separated into n sets of $n-1$ columns each, then the determinant whose $(r, s)^{\text{th}}$ element is the determinant formed from the r^{th} set and the s^{th} column of the last set will remain constant while the n sets are cyclically permuted.*

4. These theorems are seen to be of the type which almost mechanically give rise to others by the application of the law of complementaries and the law of extensible minors. On account however of this very facility of derivation space need not be occupied with the formulation of the results. It is more interesting to make use of one of the theorems to illustrate a process which renders the second of these laws still more productive.

5. The theorem is that which we have spoken of as a condensation-theorem. Let us take the simplest case, which

by the way is already well known in a different connection, namely, that which concerns the array

$$\begin{matrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3, \end{matrix}$$

and is

$$\begin{vmatrix} |a_1 b_2| & |a_1 b_3| & |a_2 b_3| \\ |c_1 d_2| & |c_1 d_3| & |c_2 d_3| \\ |e_1 f_2| & |e_1 f_3| & |e_2 f_3| \end{vmatrix} = \begin{vmatrix} |a_1 b_2 e_3| & |a_1 b_2 f_3| \\ |c_1 d_2 e_3| & |c_1 d_2 f_3| \end{vmatrix}.$$

Let us also take the simplest extensional of it, namely,

$$\begin{vmatrix} |a_1 b_2 w_4| & |a_1 b_3 w_4| & |a_2 b_3 w_4| \\ |c_1 d_2 w_4| & |c_1 d_3 w_4| & |c_2 d_3 w_4| \\ |e_1 f_2 w_4| & |e_1 f_3 w_4| & |e_2 f_3 w_4| \end{vmatrix} = w_4 \begin{vmatrix} |a_1 b_2 e_3 w_4| & |a_1 b_2 f_3 w_4| \\ |c_1 d_2 e_3 w_4| & |c_1 d_2 f_3 w_4| \end{vmatrix},$$

which besides the four w's involves the additional row of elements

$$a_4 \quad b_4 \quad c_4 \quad d_4 \quad e_4 \quad f_4,$$

each element of the determinant on the left being expressible in terms of w's and two-line minors of the enlarged array, and a similar statement holding for each element of the determinant on the right. When the elements have been expanded in this way the left-hand determinant can be partitioned in 27 determinants from each of which one of the factors

$$w_4^3, \ w_4^2 w_3, \ w_4^2 w_2, \ \ldots, \ w_4 w_1^2$$

can be separated; and consequently when similar action is taken in connection with the other determinant we have the opportunity of comparing the cofactor of one of these products of w's on the one side with the corresponding cofactor on the other. Now the interesting point is this: that while in the case of $w_4 . w_4^2$, $w_4 . w_3^2$, $w_4 . w_2^2$, $w_4 . w_1^2$ we only get a repetition of the identity with which we started, a fresh result is obtained from the other six cases. In the case of $w_4^2 w_3$ it is

$$\begin{vmatrix} |a_1 b_2| & |a_1 b_3| & |a_2 b_4| \\ |c_1 d_2| & |c_1 d_3| & |c_2 d_4| \\ |e_1 f_2| & |e_1 f_3| & |e_2 f_4| \end{vmatrix} + \begin{vmatrix} |a_1 b_2| & |a_1 b_4| & |a_2 b_3| \\ |c_1 d_2| & |c_1 d_4| & |c_2 d_3| \\ |e_1 f_2| & |e_1 f_4| & |e_2 f_3| \end{vmatrix}$$

$$= \begin{vmatrix} |a_1 b_2 e_3| & |a_1 b_2 f_4| \\ |c_1 d_2 e_3| & |c_1 d_2 f_4| \end{vmatrix} + \begin{vmatrix} |a_1 b_2 e_4| & |a_1 b_2 f_3| \\ |c_1 d_2 e_4| & |c_1 d_2 f_3| \end{vmatrix},$$

and the essential part of the rule of construction for all of them is to select for use in both determinants on the left one of the six combinations

$$12,\ 13,\ 14,\ 23,\ 24,\ 34,$$

say 23, to exclude the complementary combination 14, and to assign one of the remaining pairs of complementaries to each of the said determinants, the two full sets of indices thus being

$$23\ 21\ 34,\quad 23\ 24\ 31.$$

From these the sets for the determinants on the right-hand at once follow, namely,

$$231\ 234,\quad 234\ 231.$$

6. Treating the result of § 5 in exactly the same way as we there treated the original result we obtain

$$\begin{aligned}
&||a_5 b_2|\ |c_1 d_3|\ |e_2 f_4||+||a_1 b_2|\ |c_5 d_3|\ |e_2 f_4|| \\
+&||a_5 b_2|\ |c_1 d_4|\ |e_2 f_3||+||a_1 b_2|\ |c_5 d_4|\ |e_2 f_3|| \\
=&\begin{vmatrix} |a_1 b_2 e_3| & |a_2 b_4 f_5| \\ |c_1 d_2 e_3| & |c_2 d_4 f_5| \end{vmatrix} + \begin{vmatrix} |a_2 b_3 e_5| & |a_1 b_2 f_4| \\ |c_2 d_3 e_5| & |c_1 d_2 f_4| \end{vmatrix} \\
+&\begin{vmatrix} |a_1 b_2 e_4| & |a_2 b_3 f_5| \\ |c_1 d_2 e_4| & |c_2 d_3 f_5| \end{vmatrix} + \begin{vmatrix} |a_2 b_4 e_5| & |a_1 b_2 f_3| \\ |c_2 d_4 e_5| & |c_1 d_2 f_3| \end{vmatrix},
\end{aligned}$$

there being now involved a further additional row of elements

$$a_5\ b_5\ c_5\ d_5\ e_5\ f_5.$$

7. The process may of course be continued, but the interest culminates with the addition of the next row, for then the array of elements has become square, and the four equalities which we have arrived at form *a series of relations connecting three-line minors of the second compound of* $|a_1 b_2 c_3 d_4 e_5 f_6|$ *with two-line minors of the third compound.*

8. Each member of the series of equalities stands to the preceding one in the relation of general to particular; thus, if the index 5 be changed into 1 in the equality of § 6 there results the equality of § 5 with both of its members doubled, and there is a similar outcome when in the equality of § 5 the index 4 is changed into 3.

9. As has already been implied, there is a similar series

of equalities originating in connection with a 4-by-12 array, the fundamental one being that written above in the form

$$||a_1 b_2 c_3|\cdot|d_1 e_2 f_4|\cdot|g_1 h_3 i_4|\cdot|j_2 k_3 l_4||=\Gamma_{3,4}.$$

In that case, however, the number of determinants increases *three*-fold at each step, the first of the derivative equalities being

$$\begin{aligned}
&||a_1 b_2 c_3|\cdot|d_1 e_2 f_4|\cdot|g_1 h_3 i_4|\cdot|j_2 k_3 l_5||\\
+&||a_1 b_2 c_3|\cdot|d_1 e_2 f_4|\cdot|g_1 h_3 i_5|\cdot|j_2 k_3 l_4||\\
+&||a_1 b_2 c_3|\cdot|d_1 e_2 f_5|\cdot|g_1 h_3 i_4|\cdot|j_2 k_3 l_4||\\
=&\begin{vmatrix}|a_1b_2c_3j_4||a_1b_2c_3k_4||a_1b_2c_3l_5|\\ |d_1e_2f_3j_4||d_1e_2f_3k_4||d_1e_2f_3l_5|\\ |g_1h_2i_3j_4||g_1h_2i_3k_4||g_1h_2i_3l_5|\end{vmatrix}+\begin{vmatrix}|a_1b_2c_3j_4||a_1b_2c_3k_5||a_1b_2c_3l_4|\\ |d_1e_2f_3j_4||d_1e_2f_3k_5||d_1e_2f_3l_4|\\ |g_1h_2i_3j_4||g_1h_2i_3k_5||g_1h_2i_3l_4|\end{vmatrix}\\
&+\begin{vmatrix}|a_1b_2c_3j_5||a_1b_2c_3k_4||a_1b_2c_3l_4|\\ |d_1e_2f_3j_5||d_2e_2f_3k_4||d_1e_2f_3l_4|\\ |g_1h_2i_3j_5||g_1h_2i_3k_4||g_1h_2i_3l_4|\end{vmatrix},
\end{aligned}$$

and reverting to the original when the index 5 is changed into 4.

10. It need only be pointed out in conclusion that the series of relations (§§ 5–7) connecting the second and third compounds of $|a_1 b_2 c_3 d_4 e_5 f_6|$ may be brought into touch with the fourth compound as well, by reason of the fact that the second and fourth compounds are complementary. For example, if we return again to the simplest case of our condensation-theorem, namely,

$$\begin{vmatrix}|a_1b_2|&|a_1b_3|&|a_2b_3|\\ |c_1d_2|&|c_1d_3|&|c_2d_3|\\ |e_1f_2|&|e_1f_3|&|e_2f_3|\end{vmatrix}=\begin{vmatrix}|a_1b_2e_3|&|a_1b_2f_3|\\ |c_1d_2e_3|&|c_1d_2f_3|\end{vmatrix},$$

and take the complementary of it with respect to $|a_1 b_2 c_3 d_4 e_5 f_6|$ we obtain

$$\begin{aligned}
\begin{vmatrix}|c_3d_4e_5f_6|&|c_2d_4e_5f_6|&|c_1d_4e_5f_6|\\ |a_3b_4e_5f_6|&|a_2b_4e_5f_6|&|a_1b_4e_5f_6|\\ |a_3b_4c_5d_6|&|a_2b_4c_5d_6|&|a_1b_4c_5d_6|\end{vmatrix}&=|a_1b_2..f_6|\cdot\begin{vmatrix}|c_4d_5f_6|&|c_4d_5e_6|\\ |a_4b_5f_6|&|a_4b_5e_6|\end{vmatrix}\\
&=|a_1b_2..f_6|\cdot\begin{vmatrix}|a_4b_5e_6|&|a_4b_5f_6|\\ |c_1d_5e_6|&|c_4d_5f_6|\end{vmatrix}
\end{aligned}$$

and therefore by a reverse use of the condensation-theorem

$$= |a_1 b_2 \ldots f_6| \cdot \begin{vmatrix} |a_4 b_5| & |a_4 b_6| & |a_5 b_6| \\ |c_4 d_5| & |c_4 d_6| & |c_5 d_6| \\ |e_4 f_5| & |e_4 f_6| & |e_5 f_6| \end{vmatrix},$$

—that is to say, we have found a 3-line minor of the fourth compound of $|a_1 b_2 \ldots f_6|$ which contains as a factor a 3-line minor of the second compound, the cofactor being $|a_1 b_2 \ldots f_6|$. But the general theorem connecting a minor of a compound with a minor of the complementary compound gives us the determinant on the left

$$= |a_1 b_2 \ldots f_6|^{-2} \cdot (\text{a 12-line minor of the second compound}),$$

so that the said 12-line of the second compound is seen to contain as a factor a 3-line minor of the same compound, with the cofactor $|a_1 b_2 \ldots f_6|^3$.

Rondebosch, South Africa.

NOTE ON AN EXPRESSION OF LAMBERT'S SERIES AS A DEFINITE INTEGRAL.

By *G. H. Hardy.*

In a memoir which appeared recently in the *Acta Mathematica**, Mr. S. Wigert refers to the formula

$$(1) \quad L(z) = \sum_1^\infty \frac{1}{e^{nz}-1} = \frac{\gamma - \log z}{z} + \tfrac{1}{4} + \int_0^\infty \left(\cot \tfrac{1}{2} zt - \frac{2}{zt}\right) \frac{dt}{e^{2\pi t}-1},$$

which seems to have been given first by Cesàro†. It is supposed, of course, that the real part of z is positive. Mr. Wigert points out that the formula is false when z is complex, the difference between the left hand and right hand sides being

$$\mp \frac{2\pi i}{z} L\left(\frac{4\pi^2}{z}\right),$$

* 'Sur la série de Lambert et son application à la théorie des nombres', *Acta Mathematica*, vol. xli., 1918, pp. 197–218 (p. 209).

† 'Sur les nombres de Bernoulli et d'Euler', *Nouvelles Annales*, ser. 3, vol. vi., 1886, pp. 305–327 (p. 325). Wigert refers to Torelli, 'Sulla totalità dei numeri primi fino ad un limite assignato', *Atti della Accad. di Napoli*, ser. 2, vol. xi., 1902, pp. 1–212 (p. 188). Torelli merely quotes Cesàro, who in turn offers no proof.

according as the imaginary part of z is positive or negative; while if z is real the integral is not, in the ordinary sense, convergent.

Mr. Wigert's criticism is perfectly just. I wish however to point out that, when z is *real*, the formula is quite correct if the integral is interpreted as a 'principal value' in the sense of Cauchy. This may be shown by a fairly straightforward, though complicated, piece of contour integration, of which I need only sketch the general lines.

We consider the integral

$$\int \left\{ \frac{2ie^{zti}}{e^{zti}-1} - \frac{2}{zt(1+t^2)} \right\} \frac{dt}{e^{2\pi t}-1},$$

taken round the positive parts of the real and imaginary axes and a large quadrant of a circle with its centre at the origin. The integrand has an infinity of poles on the axes, which we exclude by small semicircles (a small quadrant in the case of the pole at the origin) in the ordinary manner. The poles are (i) a simple pole at the origin, with residue $i/2\pi$; (ii) simple poles at the points $2n\pi/z$ ($n = 1, 2, \ldots$), with residues

$$\frac{2}{z\,(e^{4n\pi^2/z}-1)};$$

(iii) simple poles at the points ni ($n = 2, 3, \ldots$), with residues

$$-\frac{i}{\pi}\left\{ \frac{1}{e^{nz}-1} + \frac{1}{nz\,(n^2-1)} \right\};$$

and (iv) a double pole at the point i, in the neighbourhood of which the integrand behaves like

$$\frac{1+\frac{3}{2}i\epsilon-\pi\epsilon^2+\ldots}{2\pi z\epsilon^2} - \frac{i}{(e^z-1)\,\pi\epsilon}+\ldots,$$

where $\epsilon = t - i$. There are no poles inside the contour.

We apply Cauchy's Theorem, *retaining only the real parts of the resulting equation*, and proceed to the limit in the obvious manner. The large quadrant contributes nothing, and the small one $\frac{1}{4}$. The real axis contributes

$$P\int_0^\infty \left\{ \cot \tfrac{1}{2}zt - \frac{2}{zt(1+t^2)} \right\} \frac{dt}{e^{2\pi t}-1},$$

which exceeds the integral in (1) by

$$\frac{2}{z}\int_0^\infty \frac{t}{1+t^2}\,\frac{dt}{e^{2\pi t}-1} = \frac{\gamma-\frac{1}{2}}{z}\,*.$$

* See Nielsen, *Gammafunktion*, pp. 85, 181.

The imaginary axis contributes

$$P\int_0^\infty \left\{\frac{e^{-zw}}{1-e^{-zw}} - \frac{1}{zw(1-w^2)}\right\} dw = -\frac{\log z}{z}.$$

Finally, the poles on the real axis (other than that at the origin) contribute nothing; those on the imaginary axis (other than those at the origin and at $t=i$) contribute

$$-\sum_2^\infty \frac{1}{e^{nz}-1} - \frac{1}{z}\sum_2^\infty \frac{1}{n(n^2-1)} = -L(z) + \frac{1}{e^z-1} - \frac{1}{4z};$$

and that at $t=i$ contributes

$$-\frac{1}{e^z-1} + \frac{3}{4z}.$$

Adding all these contributions, and equating their sum to zero, we obtain the result required.

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